Eberhard-Karls-Universität Tübingen
Prof. Hannah Markwig
Alheydis Geiger

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Introduction to Commutative Algebra and algebraic Geometry
Presence Exercise to Sheet 8

Exercise 1.
Compute $I \cap K[y,z]$ for $I = \langle x^2+y^2+1, z-x-1\rangle$.
Can you generalise the result for $J = \langle f(x,y), z-x-1\rangle$ with $f \in K[x,y]$?

Solution: Choose $>_{lp}$ with $x > y > z$. By algorithm 2.5.3 we have to compute a Gröbner basis S of I w.r.t. $>_{lp}$.

$$\begin{aligned}
f_1 &= x^2+y^2+1 \\
f_2 &= -x+z-1 \\
\mathsf{spoly}(f_1,f_2) &= -(x^2+y^2+1) - x(-x+z-1) \\
&= \\
q_2 &= z \\
r &= -y^2-1-xz+x-z(-x+z-1) \\
&= x-y^2-1-z^2+z \\
q_2 &= z-1 \\
r &= x-y^2-1-z^2+z-(-x+z-1) \\
&= -y^2-z^2+2z-2 \\
\mathsf{LM}(-y^2-z^2+2z-2) &= y^2 \\
\mathsf{LM}(f_1) &= x^2, \mathsf{LM}(f_2) = x
\end{aligned}$$

$\Rightarrow S = \{f_1, f_2, -y^2-z^2+2z-2\}$. By the product criterion this is a Gröbner basis of I.
$\Rightarrow I \cap K[y,z] = \langle -y^2-z^2+2z-2\rangle$.

Can we generalise the result?
We observe that $I \cap K[y,z] = \langle -y^2-z^2+2z-2\rangle = \langle f_1(z-1,y)\rangle$.
Does this work for any $f \in K[x,y]$?
Take $J = \langle f(x,y), -x+z-1\rangle$. Write $\mathsf{LM}(f) = x^k y^l$. w.l.o.g. we can assume $k > 0$, since otherwise J would already be given by a Gröbner basis by the product criterion. Further by our choice of monomial order this implies that $f \in K[y]$ and thus $J \cap K[y,z] = \langle f(y)\rangle$ which fits with our theory. We compute the spoly:

$$\begin{aligned}
\mathsf{spoly}(f(x,y), -x+z-1) &= -f(x,y) - \mathsf{LC}(f)x^{k-1}y^l(-x+z-1) \\
&= -\mathsf{tail}(f) - \mathsf{LC}(f)x^{k-1}y^l(z-1)
\end{aligned}$$

The leading Term of this is $\mathsf{LC}(f)x^{k-1}y^l z$. If $k-1=0$ the algorithm terminates and the output is $f(z-1,y)$. Otherwise we continue with indeterminate division with $-x+z-1$:

$$\begin{aligned}
q_2 &= \mathsf{LC}(f)x^{k-2}y^l z \\
r &= -\mathsf{LC}(f)x^{k-1}y^l(z-1) - \mathsf{tail}(f) - \mathsf{LC}(f)x^{k-2}y^l z(-x+z-1) \\
&= \mathsf{LC}(f)x^{k-1}y^l - \mathsf{tail}(f) - \mathsf{LC}(f)x^{k-2}y^l z^2 + \mathsf{LC}(f)x^{k-2}y^l z \\
q_2 &= \mathsf{LC}(f)x^{k-2}y^l z - \mathsf{LC}(f)x^{k-2}y^l \\
r &= \mathsf{LC}(f)x^{k-1}y^l - \mathsf{tail}(f) - \mathsf{LC}(f)x^{k-2}y^l z^2 + \mathsf{LC}(f)x^{k-2}y^l z + \mathsf{LC}(f)x^{k-2}y^l(-x+z-1) \\
&= -\mathsf{tail}(f) - \mathsf{LC}(f)x^{k-2}y^l z^2 + \mathsf{LC}(f)x^{k-2}y^l z + \mathsf{LC}(f)x^{k-2}y^l(z-1) \\
&= -\mathsf{tail}(f) - \mathsf{LC}(f)x^{k-2}y^l z^2 + 2\mathsf{LC}(f)x^{k-2}y^l z - \mathsf{LC}(f)x^{k-2}y^l \\
&= -\mathsf{tail}(f) - \mathsf{LC}(f)x^{k-2}y^l(z-1)^2
\end{aligned}$$

We see that continuous use of the the indeterminate division with $(-x+z-1)$ will eliminate x and lead to the output $f(z-1,y)$.