Introduction to tropical enumerative geometry

Tropical geometry can be viewed as algebraic geometry over the tropical semiring.

It works with methods from discrete mathematics (convex geometry, combinatorics) which are of intrinsic interest.

In this class, we focus completely on the discrete math - side.

Students with background knowledge in algebraic geometry can value the further motivation that tropical geometry allows a fruitful exchange of methods between algebraic geometry and discrete mathematics.

Students without such a background can follow without any trouble, they just won't have access to this additional motivation.

Tropical geometry has, besides its natural connection to algebraic geometry, also many connections to other fields of mathematics, e.g. optimization or biomathematics.

In this class, we focus on enumerative tropical geometry.

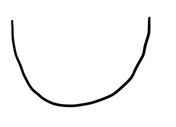
Enumerative geometry is an ancient area of mathematics, in which we ask questions about the number of geometric objects (often curves, i.e. 1-dim geometric objects) that satisfy certain conditions.

__Example__ (Apollonius' Problem)

How many circles in $\mathbb{R}^2$ are tangent to three given circles?

(Answer: 8, see Wikipedia).

Questions like this are often easy to ask, but difficult to answer, which is what makes the area of enumerative geom. a lively and active research area to this day, mostly in algebraic geometry, where we count algebraic curves satisfying conditions (i.e. solution sets of polynomials $f(x,y)$, e.g.

$y - x^2 = 0$ ).

The enumerative geometry of algebraic curves is also related to mathematical physics, e.g. string theory.

As already said, tropical geometry provides a translation from algebraic geometry to discrete mathematics.

Consequently, enumerative problems from algebraic geometry become enumerative problems in discrete maths.

Such problems are at the center of attention of this class.

Concretely, we will answer the question:

$$\boxed{\text{How many rational plane tropical curves of degree } d \text{ pass through generic given } 3d-1 \text{ points in } \mathbb{R}^2?}$$

(What such tropical plane curves are will also be studied in class, of course.)

We will prove a beautiful combinatorial formula to determine these numbers recursively.
This formula is named after the fields medaillist Kontsevich who first discovered it in connection with algebraic geometry and string theory (see the class on Gromov-Witten-theory which I gave recently and might give again in the future).

These handwritten notes are supplemented by typed notes by Renzo Cavalieri, which we will also use.

Outline

1. The tropical semiring,
   tropical polynomials,
   tropical hypersurfaces and duality
2. Algebraic curves and the
   Puiseux series
3. Abstract tropical varieties
4. Moduli spaces of rational tropical
   curves
5. Moduli spaces of stable maps
6. Kontsevich's formula

1. The tropical semiring and tropical polynomials

1.1 Def (tropical semiring)

$(\mathbb{R} \cup \{-\infty\}, \oplus, \odot)$ is called the tropical semiring, where

$x \oplus y := \max\{x, y\}$,

$x \odot y := x + y$

These operations are associative:

$(x \oplus y) \oplus z = \max\{\max\{x,y\}, z\} =$
$\max\{x,y,z\} = \max\{x, \{y,z\}\} = x \oplus (y \oplus z)$

$(x \odot y) \odot z = (x+y)+z = x+(y+z) = x \odot (y \odot z)$

distributive:

$x \odot (y \oplus z) = x + \max\{y,z\} =$
$\max\{x+y, x+z\} = x \odot y \oplus x \odot z$

commutative.

The neutral element for addition is $-\infty$.

The neutral element for multiplication is 0.

Multiplicative inverses are usual

additive inverses.

But $\nexists$ additive inverses, as the equation $x \oplus a = -\infty \iff$
$$\max\{x, a\} = -\infty$$
has no solution.

$\Rightarrow$ We cannot subtract tropically.

$(\mathbb{R} \cup \{-\infty\}, \oplus, \odot)$ satisfies all field axioms except $\exists$ additive inverses, it is called a <u>semiring</u> or <u>semifield</u>.

The tropical semiring is idempotent, i.e. $a \oplus \cdots \oplus a = a$.

The Freshman's dream holds tropically:
$$(x \oplus y)^2 = (x \oplus y) \odot (x \oplus y) =$$
$$\max\{x, y\} + \max\{x, y\} = \max\{2x, 2y\}$$
$$= x \odot x \oplus y \odot y = x^2 \oplus y^2$$

<u>Remark</u> Many authors use $(\mathbb{R} \cup \{\infty\}, \min, +)$ instead of $(\mathbb{R} \cup \{-\infty\}, \max, +)$. This is isomophic of course.

1.2 Def (tropical polynomials)

Usually, a polynomial is a finite sum of terms of the form

$a_\alpha \underline{x}^\alpha = a_\alpha \cdot x_1^{\alpha_1} \cdot \ldots \cdot x_n^{\alpha_n}$ for $\alpha \in \mathbb{N}^n$

and a_α in the ring/field of coefficients.

Tropically, we do the same:

A <u>tropical term</u> is an expression of the form $a_\alpha \odot x_1^{\alpha_1} \odot \cdots \odot x_n^{\alpha_n}$

$$= a_\alpha \odot (\underbrace{x_1 \odot \cdots \odot x_1}_{\alpha_1}) \odot \cdots \odot (\underbrace{x_n \odot \cdots \odot x_n}_{\alpha_n})$$

$$= a_\alpha + (\underbrace{x_1 + \cdots + x_1}_{\alpha_1}) + \cdots + (\underbrace{x_n + \cdots + x_n}_{\alpha_n})$$

$$= a_\alpha + \alpha_1 x_1 + \cdots + \alpha_n x_n$$

$$= a_\alpha + \langle \alpha, \underline{x} \rangle$$

(where $\langle \, , \rangle$ denotes the Euclidean scalar product on $\mathbb{R}^n$).

Viewed as function $\mathbb{R}^n \to \mathbb{R}$, a tropical term is an <u>affine-linear</u> function with <u>rational slope</u> (i.e. $\alpha \in \mathbb{N}^n$).

A <u>tropical polynomial</u> is a tropical sum of tropical terms, i.e.

$$\max_{\alpha \in \mathbb{N}^n} \{ a_\alpha + \alpha_1 x_1 + \dots + \alpha_n x_n \}$$

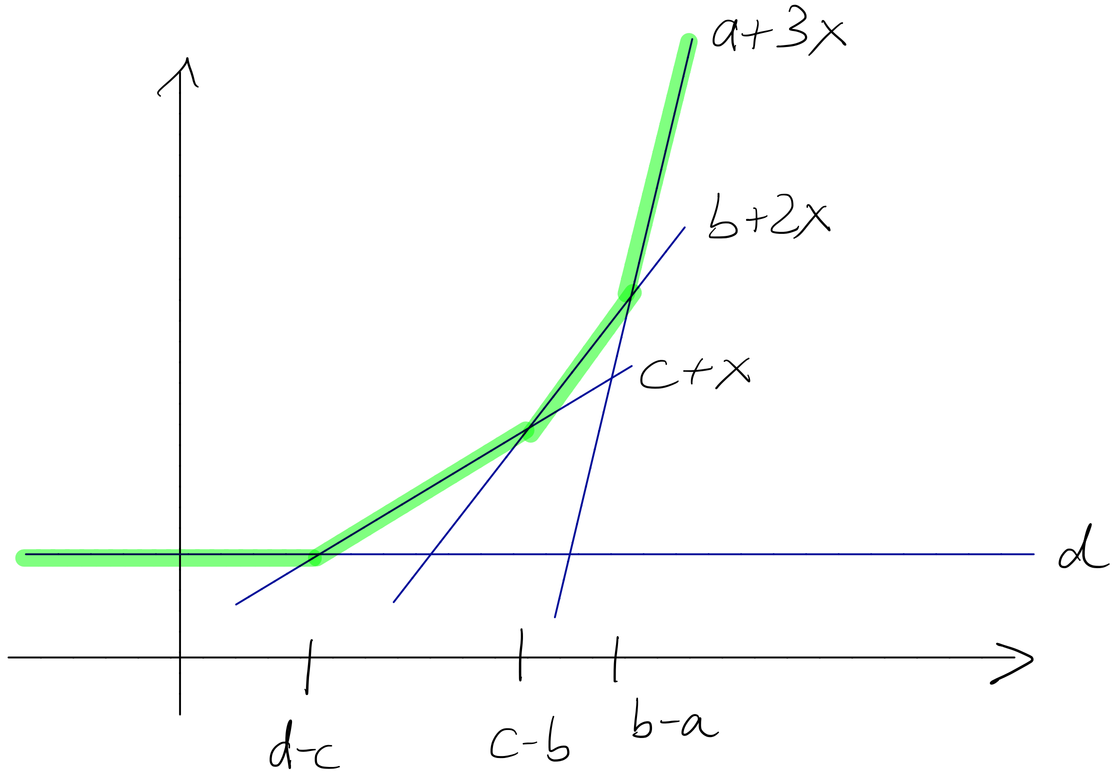
Viewed as function $\mathbb{R}^n \to \mathbb{R}$, a tropical polynomial is a piecewise affine-linear function with finitely many pieces and rational slopes, which is continuous and convex.

<u>Remark:</u> There is a difference between tropical polynomials and tropical polynomial functions. (See Questions/activities 1.1.(8) in Renzo's notes.)

<u>1.3 Example</u> (Cubic univariate polynomials)

Let $f(x) = a \odot x^3 \oplus b \odot x^2 \oplus c \odot x \oplus d$

Assume $d - c \leq c - b \leq b - a$



By our assumption, all four lines are "visible" and we have three corner loci, at $x = d-c$, $c-b$, $b-a$.

With these, we obtain a factorization of f into linear terms:

$$f = a \odot (x \oplus (d-c)) \odot (x \oplus (c-b)) \odot (x \oplus (b-a))$$

The corner loci are therefore also called the <u>zeros</u> or <u>roots</u> of f.

<u>Exercise</u>: Every univariate tropical polynomial function can uniquely be written as product of linear terms, with corner loci as roots.

<u>Remark</u>: Already in the bivariate case, there is no unique factorization, e.g.

$$(x \oplus 0) \odot (y \oplus 0) \odot (x \odot y \oplus 0)$$
$$= (x \odot y \oplus x \oplus 0) \odot (x \odot y \oplus y \oplus 0)$$

Questions/Activities 1.1 in Renzo's notes are useful now.

Tropical operation naturally appear in optimization:

Let G be a directed graph with n vertices $1, \dots, n$.
Let $d_{ij} > 0$ be the length of the edge from i to j.
Let D_G be the adjacency matrix, i.e. $d_{ii} = 0$, $d_{ij} = \infty$ if no edge exists, $D_G = (d_{ij})_{ij}$.

1.4 Prop

$-((-D_G)^{n-1})_{ij}$, where the matrix multiplication is tropical, is the length of the shortest path from i to j in G.

Proof:

Let $d_{ij}^{(r)}$ be the the length of a shortest path from i to j which takes at most r edges.

Then $d_{ij}^{(1)} = d_{ij}$.

As $d_{ij} > 0$ a shortest path can visit each vertex at most once.

In particular, it takes at most $n-1$ edges, and $d_{ij}^{(n-1)}$ is the desired length.

We have to show $d_{ij}^{(n-1)} = -((-D_G)^{n-1})_{ij}$

For $r \geq 2$ we have

$d_{ij}^{(r)} = \min_k \{ d_{ik}^{(r-1)} + d_{kj} \} =$

$- \max_k \{ -d_{ik}^{(r-1)} - d_{kj} \} =$

$$-\left(\left(-d_{i_1}\right)^{(r-1)} \odot (-d_{1j}) \oplus \cdots \oplus \left((-d_{in})^{(r-1)} \odot (-d_{nj})\right)\right) =$$

$$-\left(-d_{i1}^{(r-1)}, \ldots, -d_{in}^{(r-1)}\right) \odot \begin{pmatrix} -d_{1j} \\ \vdots \\ -d_{nj} \end{pmatrix}$$

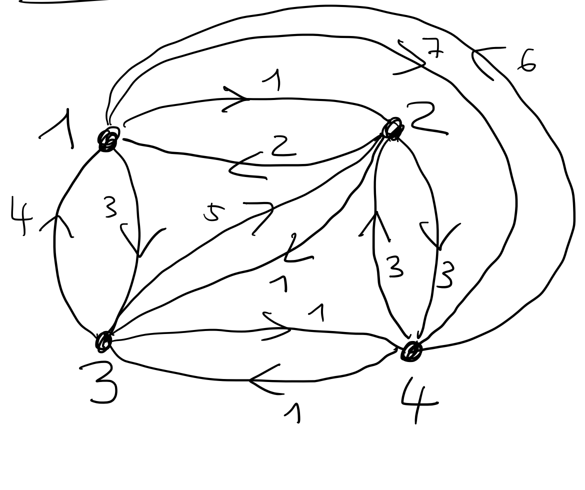
By induction, we conclude that

$$d_{ij}^{(r)} = -\left((-D_G)^r\right)_{ij}.$$

$\square$

Thus finding shortest paths on graphs (optimization) is equivalent to taking tropical powers of matrices.

Example



$$-D_G = \begin{pmatrix} 0 & -1 & -3 & -7 \\ -2 & 0 & -1 & -3 \\ -4 & -5 & 0 & -1 \\ -6 & -3 & -1 & 0 \end{pmatrix}$$

$$(-D_G)^2 = \begin{pmatrix} 0 & -1 & -2 & -4 \\ -2 & 0 & -1 & -2 \\ -4 & -4 & 0 & -1 \\ -5 & -3 & -1 & 0 \end{pmatrix}$$

$$(-D_G)^3 = \begin{pmatrix} 0 & -1 & -2 & -3 \\ -2 & 0 & -1 & -2 \\ -4 & -4 & 0 & -1 \\ -5 & -3 & -1 & 0 \end{pmatrix}$$

For example, to go from 1 to 4:

- directly: 7
- in two steps: $1 \to 3 \to 4$: 4
- in three steps: $1 \to 2 \to 3 \to 4$: 3

The tropical computation reflects the following usual computation:
Let t be a variable and consider the matrix

$$\begin{pmatrix} 1 & t & t^3 & t^7 \\ t^2 & 1 & t & t^3 \\ t^4 & t^5 & 1 & t \\ t^6 & t^3 & t & 1 \end{pmatrix}$$

Then minus the (smallest) exponents of the (usual) powers of this matrix equals the tropical matrices above.

1.5 Def (tropical hypersurface)

Let f be a tropical polynomial in n variables. Then

$$V(f) = \{x \in \mathbb{R}^n \mid \text{the maximum of } f \text{ is attained at least by two monomials}\}$$

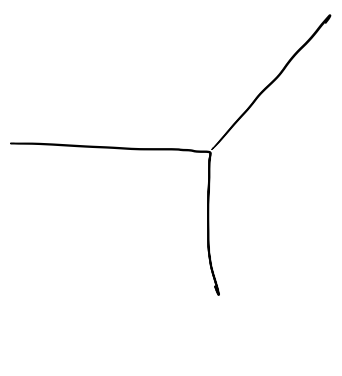
$=$ the corner locus of the piecewise linear function f

is called the <u>tropical hypersurface</u> defined by f. If $n=2$, we call it a <u>plane curve</u>.

Example:

$f = x \oplus y \oplus 0 = \max\{x, y, 0\}$

$V(f) =$



a tropical line

Questions/activities 1.2 in Renzo's notes are useful now.

To describe the structure of tropical hypersurfaces better, we need to introduce a bit of convex geometry.
We will not be very formal or detailed with this, but as the subject is intuitively accessible, this should be no harm.

<u>1.6 Def</u> (convex hulls, polytops)
$X \subset \mathbb{R}^n$ is convex, if $\forall\, u, v \in X$
$\forall\, 0 \leq \lambda \leq 1 \quad \lambda u + (1-\lambda) v \in X$,
i.e. the line segment connecting u and v is in X.
The <u>convex hull</u> $\operatorname{conv}(U)$ of $U \subset \mathbb{R}^n$ is the smallest convex set containing U. If $U = \{u_1, \dots, u_r\}$ is finite,

$$\operatorname{conv}(U) = \left\{ \sum_{i=1}^{r} \lambda_i u_i \;\middle|\; 0 \leq \lambda_i \leq 1,\ \sum \lambda_i = 1 \right\}$$

is called a <u>polytop</u>.
If $U \subset \mathbb{Z}^n$ is finite, $\operatorname{conv}(U)$ is called a <u>lattice polytope</u>.

A cone is the positive hull of finitely many vectors in $\mathbb{R}^n$:

$$C = \mathrm{cone}(v_1, \ldots, v_r) := \left\{ \sum_{i=1}^{r} \lambda_i v_i \;\middle|\; \lambda_i \geq 0 \right\}$$

not convex — a polytope — convex — a cone

If $n=2$, a polytope is called polygon.

A cone is strictly convex if it contains no subspace of positive dimension.

strictly convex — not strictly convex

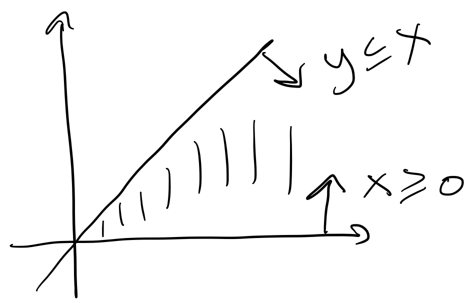
Remark:

Each cone $C = \mathrm{cone}(v_1, \ldots, v_r)$ can be given by inequalities, i.e. of the form $C = \{ x \mid Ax \geq 0 \}$ for some matrix A.

E.g. $C = \mathrm{cone}\left(\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}\right) =$

$$\left\{ \begin{pmatrix}x\\y\end{pmatrix} \;\middle|\; x \geq 0,\; y \leq x \right\}$$

$$= \left\{ \begin{pmatrix}x\\y\end{pmatrix} \;\middle|\; \begin{pmatrix}1 & 0\\1 & -1\end{pmatrix} \begin{pmatrix}x\\y\end{pmatrix} \geq 0 \right\}$$



1.7 Def (faces)

A *face* of a cone C is given by a linear functional in $(\mathbb{R}^n)^\vee$:

Let $w \in (\mathbb{R}^n)^\vee$ s. th. $w \cdot y \le 0$ $\forall\, y \in C$, then

$$\mathrm{face}_w(C) := \{ x \in C \mid w \cdot x = 0 \}$$

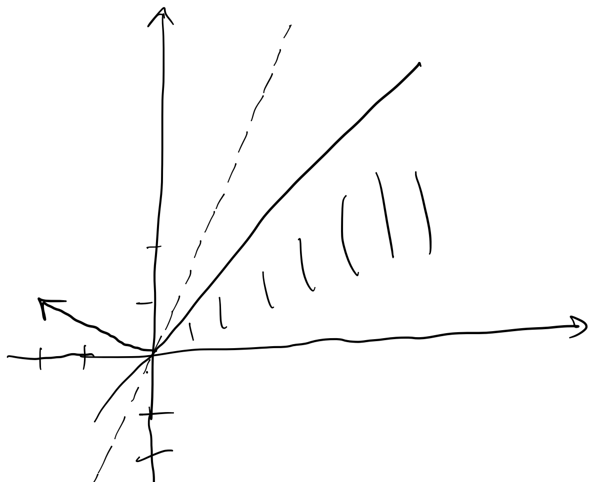
Setting $w = 0$, C is a face of itself by definition.

Example

$C = \mathrm{Cone}\left(\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}\right)$

$w = \begin{pmatrix}-2\\1\end{pmatrix}$ satisfies $w \cdot \begin{pmatrix}x\\y\end{pmatrix} \le 0$ $\forall \begin{pmatrix}x\\y\end{pmatrix} \in C$.

$\mathrm{face}_w(C) = \{0\}$



$w = \begin{pmatrix}-1\\0\end{pmatrix}$ ''

$\mathrm{face}_w(C) = \{0\}$

$w = \begin{pmatrix}0\\-1\end{pmatrix}$, $\mathrm{face}_w(C) = \mathrm{Cone}\begin{pmatrix}1\\0\end{pmatrix}$

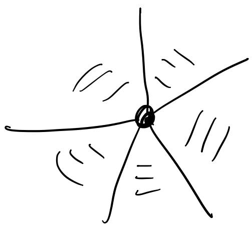
$w = \begin{pmatrix}-1\\1\end{pmatrix}$, $\mathrm{face}_w(C) = \mathrm{Cone}\begin{pmatrix}1\\1\end{pmatrix}$

Analogously to 1.7, we define faces of polytops, only here we allow affine functionals (i.e. linear with a shift).

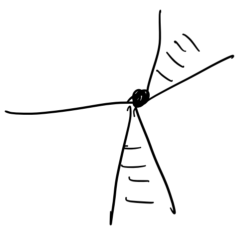
The <u>dimension</u> of a cone or convex set is the dimension of the smallest affine subspace it is contained in.

A face of maximal dimension which is not C is a <u>facet</u>. A face of dim 0 is a vertex, a face of dim 1 in a cone is a ray.

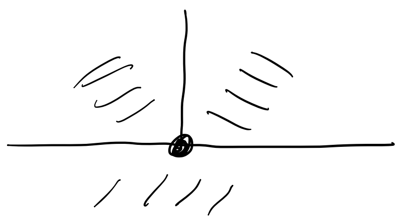
<u>1.8 Def</u> (Fan)
A <u>fan</u> is a set of cones in $\mathbb{R}^n$, s. th. the intersection of two is a face of both, and such that all faces are contained.



a fan



a fan



not a fan

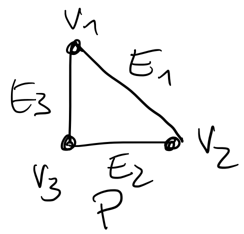
1.9 Def (outer normal fan)

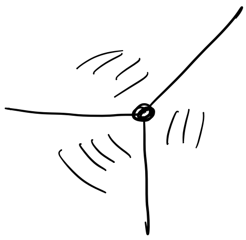
Let $P \subset \mathbb{R}^n$ be a polytope.
The <u>outer normal fan</u> $\mathcal{N}_P$ consists of the cones

$$\mathcal{N}_P(F) = \underline{\{ w \in (\mathbb{R}^n)^\vee \mid \mathrm{face}_w(P) = F \}}$$

where F is a face of P.

<u>Example:</u>





$\mathcal{N}_P$ has 7 cones:

$\{0\} = \mathcal{N}_P(P)$

$\mathrm{Cone}\begin{pmatrix}1\\1\end{pmatrix} = \mathcal{N}_P(E_1)$

$\mathrm{Cone}\begin{pmatrix}0\\-1\end{pmatrix} = \mathcal{N}_P(E_2)$, $\mathrm{Cone}\begin{pmatrix}-1\\0\end{pmatrix} = \mathcal{N}_P(E_3)$,

$\mathrm{Cone}\left(\begin{pmatrix}-1\\0\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}\right) = \mathcal{N}_P(v_1)$,

$\mathrm{Cone}\left(\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}0\\-1\end{pmatrix}\right) = \mathcal{N}_P(v_2)$, $\mathrm{Cone}\left(\begin{pmatrix}-1\\0\end{pmatrix}, \begin{pmatrix}0\\-1\end{pmatrix}\right) = \mathcal{N}_P(v_3)$

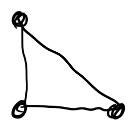
1.10 Def

The <u>Newton polytope</u> of a (tropical) polynomial $f = \sum_{\alpha \in \mathbb{N}^n} a_\alpha x^\alpha \in K[x_1, \dots, x_n]$

is $\mathrm{Newt}(f) := \mathrm{Conv}(\alpha \mid a_\alpha \neq 0) \subset \mathbb{R}^n$

resp. $a_\alpha \neq -\infty$ if tropical

<u>Example:</u> $f = x + y + 1$



1.11 Def (marked polytope and subdivision)

Let $Q \subset \mathbb{R}^n$ be a lattice polytope and $\mathcal{A} = Q \cap \mathbb{Z}^n$ the lattice points of Q.

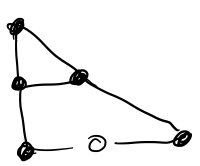
$(Q, \mathcal{A}')$ is a marked polytope if $\mathcal{A}'$ contains the vertices of Q.

A marked subdivision of Q is a set $\{(Q_i, \mathcal{A}_i) \mid i = 1, \dots, k\}$ s. th.

1) $(Q_i, \mathcal{A}_i)$ is a marked polytope
2) $Q = \bigcup_{i=1}^{k} Q_i$ is a subdivision of Q, i.e. $Q_i \cap Q_j$ is a face (possibly empty) of Q_i and Q_j
3) $\mathcal{A}_i \subset \mathcal{A} \quad \forall i$
4) $\mathcal{A}_i \cap (Q_i \cap Q_j) = \mathcal{A}_j \cap (Q_i \cap Q_j)$

Examples:

We draw marked points (in $\mathcal{A}_i$) black



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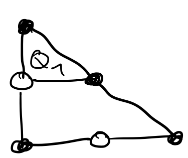

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a vertex of Q_1 is not marked

By 4), marked subdivisions can be drawn like this.

Using a height function (e.g. by defining the coefficient of a tropical polynomial to be the height), we can define a marked subdivision, the so-called <u>dual Newton subdivision</u>, by projection of upper faces, see Def. 1.5 and the paragraph above in Renzo's notes.

Read also example 1.2 and think about Questions/activities 1.3.

Read Theorem 1.1, it states the <u>duality</u> of tropical hypersurfaces and the dual Newton subdivision.

We include more ideas on the proof here:

1) Assume first that $a_\alpha = 0 \ \forall \alpha$, i.e. $f = \max_\alpha \{\alpha \cdot x\}$.

The top-dim cones of $\mathcal{N}_{\mathrm{Newt}(f)}$ correspond to the vertices, the cones of codimension 1 to the edges.

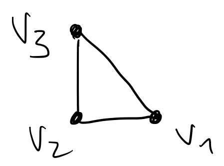
An edge E connects two vertices corresponding to α_1 and α_2.

Then $\mathcal{N}_{Newt(f)}(E)$ is contained in the hyperplane whose normal vector is E.

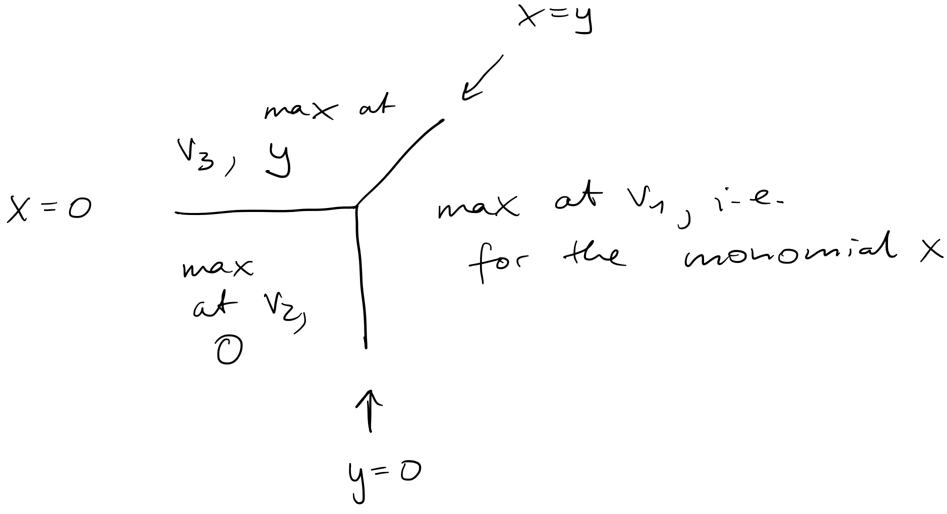
This hyperplane is given by the equation $\alpha_1 \cdot X = \alpha_2 \cdot X$.

$\mathcal{N}_{Newt(f)}(E)$ is precisely the subset of this hyperplane for which the maximum is attained at $\alpha_1 \cdot X = \alpha_2 \cdot X$.

<u>Example</u>:



$x \oplus y \oplus 0$



2) If not $a_\alpha = 0$:

Set $\tilde{f} = \sum t^{a_\alpha} \underline{x}^\alpha$ (possibly a polynomial with real exponents, but that does not make any change here), then the Newton polytope of $\tilde{f}$ is what we project to obtain the subdivision.

By 1), the tropical hypersurface $V(\tilde{f})$ is the codim-1-skeleton of $\mathcal{N}_{Newt(\tilde{f})}$. $V(f) = V(\tilde{f}) \cap \{t=1\}$

A monomial of f yields a vertex of the upper hull of $Newt(\tilde{f})$ (which we project to obtain the subdivision) $\Longleftrightarrow$ $\exists$ top-dim cone in $\mathcal{N}_{Newt(\tilde{f})}$ spanned by vectors for which the t-coordinate is positive $\Leftrightarrow$ the intersection with $\{t=1\}$ produces a component of $\mathbb{R}^n \setminus V(f)$.

This explains the duality
$\{$vertices of the subdivision$\}$ $\longleftrightarrow$
$\{$components of $\mathbb{R}^n \setminus V(f)\}$

With this, we obtain

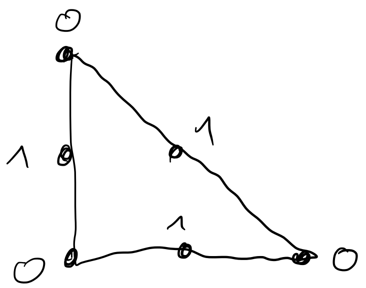
$\{$edges in the dual subdivision$\} \hookleftarrow$
$\{$edges of $V(f)$ (separating two connected components of $\mathbb{R}^n \setminus V(f)$)$\}$

and so on. $\square$

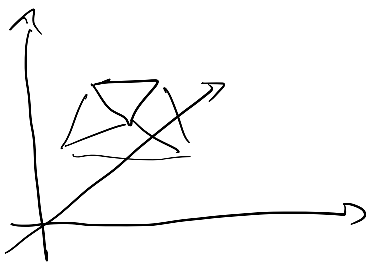
Example

Using duality, one can draw tropical plane curves quickly:

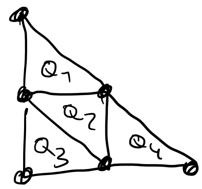
Let $f = 0 \oplus 1 \odot x \oplus x^2 \oplus 1 \odot y \oplus 1 \odot x \odot y \oplus y^2$



Newt(f)



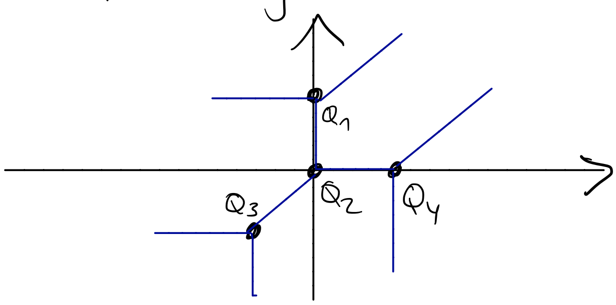
to project



subdivision

The vertex dual to Q_1 satisfies:

Q_1: $2y = 1+y = 1+x+y \Rightarrow x=0, y=1$
Q_2: $1+x = 1+x+y = 1+y \Rightarrow x=0, y=0$
Q_3: $0 = 1+x = 1+y \Rightarrow x=-1, y=-1$
Q_4: $2x = 1+x+y = 1+x \Rightarrow y=0, x=1$

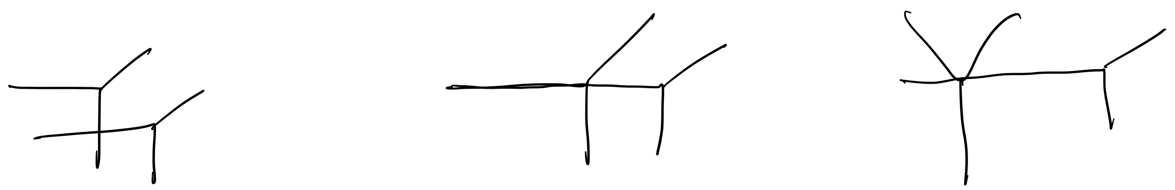


Read Def 1.6, 1.7 and Theorem 1.2 in Renzo's notes.
Questions / Activities 1.4 are useful now.

1.12 Def (deg d)
We say a tropical plane curve has **degree** d if it is dual to the polygon $\mathrm{Conv}((0,0),(d,0),(0,d))$

1.13 Def (transversal intersection and intersection multiplicity)

Two tropical plane curves $V(f)$ and $V(g)$ intersect transversally if they intersect at finitely many points which are all interior points of edges of both.

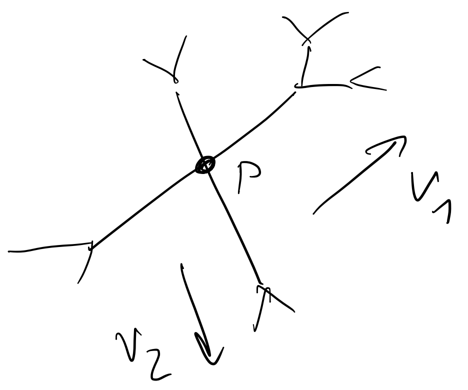


transversal   not   not

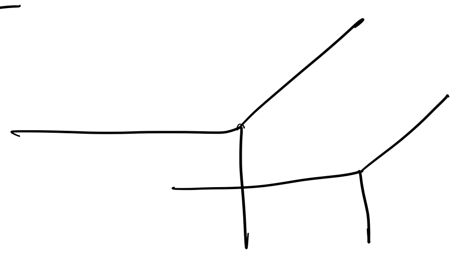
Let $p \in V(f) \cap V(g)$ be transversal.
Let w_1 be the weight of the edge e_1 of $V(f)$ in which

p is , and v_1 its direction,
and analogously for e_2.
Then we define the intersection multiplicity of $V(f)$ and $V(g)$ at p to be

$$\mathrm{mult}_p\,(V(f), V(g)) = w_1 \cdot w_2 \cdot |\det(v_1, v_2)|$$



Ex:



$$1 \cdot 1 \cdot \left|\det\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\right| = 1$$

Two lines intersect in a point with multiplicity 1.

1.14 Theorem (Bézout)

A tropical plane curve of degree d and a tropical plane curve of deg e which intersect transversally intersect in $d \cdot e$ points, counted with multiplicity.

Proof: Exercise.

2. Algebraic curves and the Puiseux series

The goal of this section is to see that tropical plane curves (or, more generally, tropical hypersurfaces) are really shadows of algebraic plane curves (hypersurfaces).

This provides an additional motivation for their study.

For those who are acquainted with algebraic geometry, you know that we like to work with algebraically closed fields.

For the others, you know that polynomials "have more solutions" over $\mathbb{C}$ (which is algebraically closed): e.g. $x^2+1=0$ has no solutions over $\mathbb{R}$ but 2 over $\mathbb{C}$.

For that reason, our first step is to define an interesting new algebraically closed field in which we can study zeros of polynomials.

Read the beginning of chapter 2, Def 2.1 of Renzo's notes. Questions/ Activities 2.1 are useful.

Def $\mathcal{R} = \mathrm{val}^{-1}(\mathbb{R}_{\geq 0}) \subset \mathbb{C}\{\{t\}\}$, $m = \mathrm{val}^{-1}(\mathbb{R}_{>0}) \subset \mathbb{C}\{\{t\}\}$

2.1 Theorem

The field $K = \mathbb{C}\{\{t\}\}$ of Puiseux series is algebraically closed.

Proof:

Let $F = \sum_{i=0}^{n} c_i x^i \in K[x]$.

We have to show: $\exists y \in K$: $F(y) = 0$.

We will describe an algorithm

which constructs y term by term.

First, we show that we can assume the following properties for F:

1) $\mathrm{val}(c_i) \geq 0 \ \forall i$
2) $\exists j : \mathrm{val}(c_j) = 0$
3) $c_0 \neq 0$
4) $\mathrm{val}(c_0) > 0$

Let $\alpha = \min\{\mathrm{val}(c_i)\}$, the multiplication of F with $t^{-\alpha}$ does not change the existence of a zero, thus we can assume 1) and 2).

If $c_0 = 0$, $y = 0$ is a zero, so we can assume 3).

Assume F satisfies 1)-3), but not 4).

If $\mathrm{val}(c_n) > 0$, let

$$G(x) = x^n \cdot F\left(\frac{1}{x}\right) = \sum_{i=1}^{n} c_{n-i} \, x^i .$$

G satisfies 1)-4), and if

$G(y') = 0$ then $F(\frac{1}{y'}) = 0$, so it is sufficient to construct a zero for G.

If $val(c_0) = val(c_n) = 0$, consider $f = \overline{F} \in \mathbb{C}[x]$ the image of F under the quotient map

$$R[x] \longrightarrow {}^{R}\!/_{m}[x] = \mathbb{C}[x].$$

f is not constant, as $val(c_n) = 0$.

As $\mathbb{C}$ is algebraically closed $\exists$ $\lambda : f(\lambda) = 0$.

Let $\tilde{F}(x) = F(x+\lambda) =$

$c_0 + c_1(x+\lambda) + c_2(x+\lambda)^2 + \dots + c_n(x+\lambda)^n$

$$= \sum_{i=0}^{n} \left(\sum_{j=i}^{n} c_j \binom{j}{i} \lambda^{j-i} \right) x^i.$$

$\tilde{F}(x)$ has the constant term

$\tilde{F}(0) = F(\lambda) = f(\lambda) +$ terms of higher valuation

$= 0 +$ terms of higher valuation

The highest term of $\tilde{F}$ is c_n of valuation 0.

Thus we can assume 1)–4) for $\tilde{F}$, and if we find y' with $\tilde{F}(y')=0$, then $F(y'+\lambda)=0$, so it is sufficient to construct a zero of $\tilde{F}$.

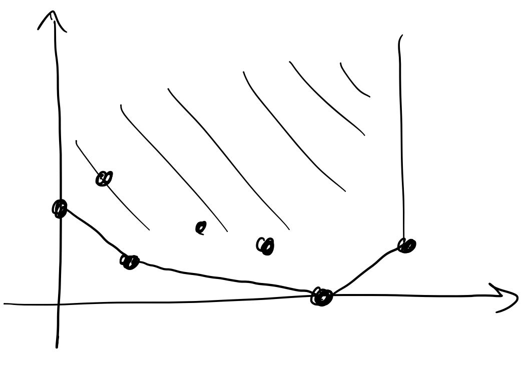
Thus, we can now assume F satisfies 1)–4).

We construct a sequence of polynomials $F_i = \sum_{j=0}^{n} c_j^i x^j$

which all satisfy 1)–4).

Set $F_0 := F$.

Consider $\mathrm{conv}\left((k,j) \,/\, \mathrm{val}(c_k^i) \leq j \right)$



We know $\mathrm{val}(c_0^i)>0$,

$\exists\, k: \mathrm{val}(c_k^i)=0$,

thus $\exists$ edge

of negative slope connecting $(0, \mathrm{val}(c_0^i))$ with another vertex $(k_i, \mathrm{val}(c_{k_i}^i))$.

Set $w_i = \dfrac{\mathrm{val}(c_0^i) - \mathrm{val}(c_{k_i}^i)}{k_i}$

and consider

$$F_i(t^{w_i} \cdot x) = \sum_{j=0}^{n} c_j^i (t^{w_i} x)^j$$

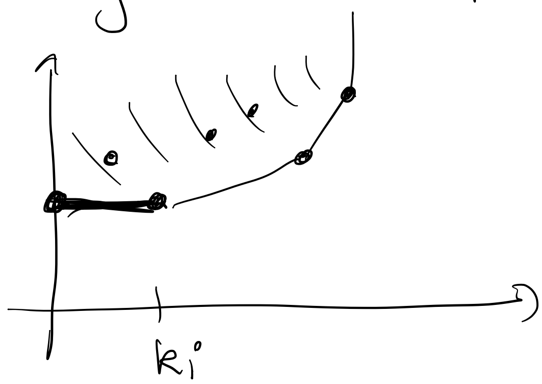
Then the valuation of the constant coefficient is $\mathrm{val}(c_0^i)$, and of the k_i th:

$$\mathrm{val}(c_{k_i}^i t^{w_i k_i}) = \mathrm{val}(c_{k_i}^i t^{\mathrm{val}\, c_0^i - \mathrm{val}\, c_{k_i}^i})$$

$$= \mathrm{val}(c_0^i),$$

all other coefficients have higher valuation.

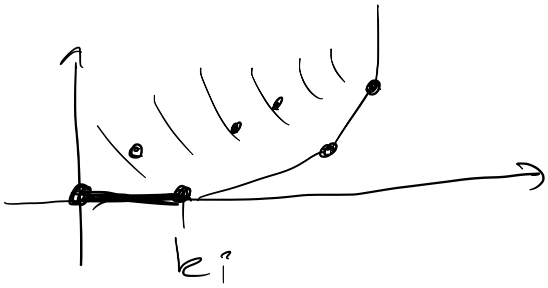
This operation "evens the edge with negative slope":



Next, consider

$$t^{-\mathrm{val}(c_0^i)} \cdot F_i(t^{w_i} x),$$

this "moves it down":



Let $f_i = \overline{t^{-val(c_0^i)} F_i(t^{w_i}x)} \in \mathbb{C}[x]$

Then $\deg(f_i) = k_i$, $f_i(0) \neq 0$.

As $\mathbb{C}$ is algebraically closed, $\exists\, d_i \in \mathbb{C}$: $f_i(d_i) = 0$. Let r_{i+1} be the multiplicity of this zero, i.e.

$$f_i = (x-d_i)^{r_{i+1}} \cdot g_i(x), \quad g_i(d_i) \neq 0.$$

Set $F_{i+1}(x) = t^{-val(c_0^i)} F_i(t^{w_i}(x+d_i))$

$$= t^{-val(c_0^i)} \cdot \sum_{l=0}^{n} c_l^i \left(t^{w_i}(x+d_i)\right)^l =$$

$$t^{-val(c_0^i)} \cdot \sum_{l=0}^{n} c_l^i \cdot \sum_{j=0}^{l} \binom{l}{j} t^{l \cdot w_i} x^j d_i^{l-j}$$

$$= \sum_{j=0}^{n} \left(\sum_{l=j}^{n} c_l^i\, t^{l \cdot w_i - val\, c_0^i} \binom{l}{j} d_i^{l-j} \right) x^j$$

$$=: \sum_{j=0}^{n} c_j^{i+1} x^j$$

Consider again $f_i = \overline{t^{-\mathrm{val}\, c_0^i} F_i(t^{w_i} x)}$

$$= \overline{\sum_{j=0}^{n} c_j^i \, t^{j \cdot w_i - \mathrm{val}\, c_0^i} \, x^j}$$

$$= \sum_{\ell \mid \ell \cdot w_i - \mathrm{val}\, c_0^i + \mathrm{val}\, c_\ell^i = 0} \overline{c_\ell^i} \, x^\ell$$

We have $\frac{1}{j!} \frac{\partial^j f_i}{(\partial x)^j} (\lambda_i) =$

$$\frac{1}{j!} \cdot \sum_{\ell \mid \ell \cdot w_i - \mathrm{val}\, c_0^i + \mathrm{val}\, c_\ell^i = 0} \overline{c_\ell^i} \frac{\ell!}{(\ell - j)!} \lambda_i^{\ell - j} =$$

$$\sum_{\ell \mid \ell \cdot w_i - \mathrm{val}\, c_0^i + \mathrm{val}\, c_\ell^i = 0} \binom{\ell}{j} \overline{c_\ell^i} \, \lambda_i^{\ell - j} = \overline{c_j^{i+1}}$$

As λ_i is a zero of mult r_{i+1},

$$\frac{\partial^j f_i}{(\partial x)^j} (\lambda_i) = \begin{cases} 0 & \forall\ 0 \le j < r_{i+1} \\ \neq 0 & j = r_{i+1} \end{cases}$$

$$\Rightarrow \quad \mathrm{val}\, (c_j^{i+1}) \begin{cases} > 0 & 0 \le j < r_{i+1} \\ = 0 & j = r_{i+1} \end{cases}$$

If $c_0^{i+1} = 0$, $X = 0$ is a zero of F_{i+1}, $d_i t^{w_i}$ a zero of F_i and

$\sum_{j=0}^{i} d_j \, t^{w_0 + \dots + w_j}$ a zero of F_0.

Thus we can assume $c_0^{i+1} \neq 0$ and then F_{i+1} satisfies 1) – 4) and we continue the construction.

As $\operatorname{val}(c_{r_{i+1}}^{i+1}) = 0$ we know $k_{i+1} \leq r_{i+1}$, and as r_{i+1} is the multiplicity of a zero of f_i of deg k_i, also $r_{i+1} \leq k_i$

As n is finite, k_i can get smaller only finitely many times

$\Rightarrow \exists \; k \in \{1, \dots, n\}, \; m \in \mathbb{N}$:

$k_i = k \quad \forall i \geq m$, $\quad r_i = k \quad \forall i > m$

$\Rightarrow f_i = \mu_i \cdot (X - d_i)^k \quad \forall i > m$

and some $\mu_i \in \mathbb{C}$.

Let N_i s.th. $c_j^i \in \mathbb{C}((t^{\frac{1}{N_i}}))$

$\forall\ 0 \le j \le n$.

As $F_{i+1}(x) = t^{-\mathrm{val}\, c_0^i} F_i(t^{w_i}(x+d_i))$

$\frac{1}{N_{i+1}}$ is the common denominator of

$\frac{1}{N_i}$ and w_i.

<u>Claim</u>: $N_{i+1} = N_i \quad \forall\ i > m$.

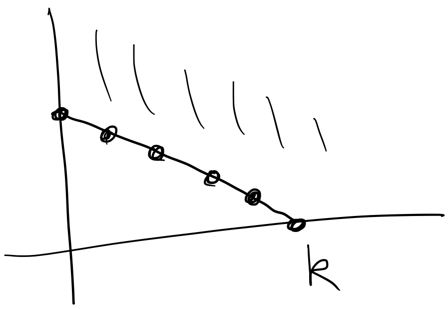
We have $w_i = \frac{\mathrm{val}\, c_0^i}{k}$, thus it

is sufficient to see $\mathrm{val}(c_0^i) \in \frac{k}{N_i} \cdot \mathbb{Z}$

$\forall\ i > m$.

As $f_i = \mu_i (x - d_i)^k$ we have

$\mathrm{val}(c_j^i) = \frac{k-j}{k} \cdot \mathrm{val}(c_0^i)$



in particular for $j = k-1$:

$\mathrm{val}(c_{k-1}^i) = \frac{1}{k} \cdot \mathrm{val}(c_0^i)$

But $\mathrm{val}(c_{k-1}^i) \in \frac{1}{N_i} \mathbb{Z} \Rightarrow$

$\frac{1}{k} \mathrm{val}(c_0^i) \in \frac{1}{N_i} \mathbb{Z} \Rightarrow$

$\mathrm{val}(c_0^i) \in \frac{k}{N_i}\mathbb{Z} \implies N_{i+1} = N_i \ \forall i > m.$

Let $y_i = \sum_{j=0}^{i} \lambda_j t^{w_0 + \dots + w_j} \in \mathbb{C}((t^{\frac{1}{N_{i+1}}}))$

as $N_{i+1} = N_i \quad \forall i > m \quad \exists N$ s.th.

$y_i \in \mathbb{C}((t^{\frac{1}{N}})) \ \forall i$, such the

limit $y = \sum_{j \geq 0} \lambda_j t^{w_0 + \dots + w_j} \in \mathbb{C}((t^{\frac{1}{N}}))$

is a Puiseux series.

It remains to see $F(y) = 0$.

Let $z_i = \sum_{j \geq i} \lambda_j t^{w_i + \dots + w_j}$, then

$y = y_{i-1} + t^{w_0 + \dots + w_{i-1}} \cdot z_i$ for $i > 0$.

We have $F_i(z_i) = t^{\mathrm{val}(c_0^i)} F_{i+1}(z_{i+1})$

As $z_0 = y$ we have

$\mathrm{val}(F(y)) = \sum_{j=0}^{i} \mathrm{val}(c_0^j) + \mathrm{val}(F_{i+1}(z_{i+1}))$

$> \sum_{j=0}^{i} \mathrm{val}(c_0^j) \quad \forall i > 0.$

As $\mathrm{val}(c_0^{\gamma}) \in \frac{1}{N}\mathbb{N}$, we can conclude $\mathrm{val}(F(y)) = \infty \Rightarrow$

$F(y) = 0$. $\square$

2.2 Def (Tropicalization)

Let $K = \mathbb{C}\{\{t\}\}$.

We define the tropicalization map

$$\mathrm{Trop}: (K^*)^n \longrightarrow \mathbb{R}^n:$$

$$(x_1, \dots, x_n) \longmapsto (-\mathrm{val}\, x_1, \dots, -\mathrm{val}\, x_n)$$

Exercise:

Compute $\mathrm{Trop}(L)$ for a Line $L \subset (K^*)^2$.

2.3 Def (tropicalization of polynomials)

Let $f = \sum_{\alpha \in \mathbb{N}^n} c_\alpha x_1^{\alpha_1} \cdots x_n^{\alpha_n}$

$\in \mathbb{C}\{\{t\}\}[x_1, \dots, x_n]$, then

$$\mathrm{Trop}(f) := \max_{\alpha}\{-\mathrm{val}(c_\alpha) + \alpha \cdot x\}$$

is the tropicalization of f.

2.4 Def (Hypersurface, plane curve)

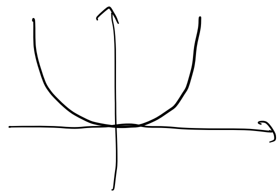
Let K be any (algebraically closed) field.

Let $f \in K[x_1, \dots, x_n]$.

The <u>hypersurface</u> of f is

$$V(f) = \{ x \in K^n \mid f(x) = 0 \}.$$

If $n=2$, we call $V(f)$ a <u>plane curve</u>.

<u>Example</u>: $V(y-x^2) =$ 

2.5 Theorem (Kapranov, see 2.1 in Renzo's notes)

Let $f = \sum_{\alpha \in \mathbb{N}^n} c_\alpha x_1^{\alpha_1} \cdots x_n^{\alpha_n}$

$\in \mathbb{C}\{\{t\}\}[x_1, \dots, x_n]$

Then

$$\mathrm{Trop}(V(f) \cap (K^*)^n) = V(\mathrm{Trop}(f))$$

(where we take the closure in the

Euclidean topology).

Proof, part I:

"$\subset$" Let $x \in V(f) \cap (K^*)^n$

$\Rightarrow \sum c_\alpha x_1^{\alpha_1} \cdots x_n^{\alpha_n} = 0$

Let $x_1 = a_1 t^{-w_1} + \cdots$, $x_n = a_n t^{-w_n} + \cdots$

Then $-\operatorname{val} x_i = w_i$.

We have to show, $w_i \in V(\operatorname{Trop} f)$,

i.e. $\max\{-\operatorname{val} c_\alpha + \alpha \cdot w\}$ is attained at least twice.

$0 = f(x) = \sum c_\alpha (a_1 t^{-w_1} + \cdots)^{\alpha_1} \cdots (a_n t^{-w_n} + \cdots)^{\alpha_n}$

The lowest order of a summand is

$\operatorname{val}(c_\alpha) - w_1\alpha_1 - \cdots - w_n\alpha_n$.

The lowest order of the whole sum is

$\min\{\operatorname{val}(c_\alpha) - w_1\alpha_1 - \cdots - w_n\alpha_n\}$

As the sum is 0, the terms cancel away, in particular the terms of lowest order cancel away, in particular there must be at least two terms of lowest order

$\Rightarrow \min\{\mathrm{val}(c_\alpha) - w_1\alpha_1 - \dots - w_n\alpha_n\}$

is attained at least twice

$\Rightarrow \max\{-\mathrm{val}(c_\alpha) + w\cdot\alpha\}$ is attained at least twice. $\square$

2.6 Lemma

Let k be any field, $g \in k[x_1, \dots, x_n]$.

g has at least two terms $\Leftrightarrow$ g has a zero in $(k^*)^n$.

The proof is easy for those who are familiar with algebraic geometry, those who aren't I would like to ask to just believe the statement for now.

2.7 Def (initial forms)

Let $f = \sum_{\alpha \in \mathbb{N}^n} c_\alpha x_1^{\alpha_1} \cdots x_n^{\alpha_n}$

$\in \mathbb{C}\{\{t\}\}[x_1, \dots, x_n]$

Let $w \in \mathbb{R}^n$, $W = \mathrm{Trop}(f)(w)$, set $\mathrm{in}_w f = \overline{t^W \sum_\alpha c_\alpha t^{-w\cdot\alpha} x^\alpha} \in \mathbb{C}[\underline{x}]$

$\mathrm{in}_w f$ is called the <u>initial form</u> of f w.r.t. w.

We have $\mathrm{in}_w(f) = \sum_{\alpha \mid -\mathrm{val}(c_\alpha)+w\cdot\alpha = W} \overline{c_\alpha t^{-\mathrm{val}\, c_\alpha}}\, x^\alpha$

$= \overline{t^W f(t^{-w_1}x_1, \dots, t^{-w_n}x_n)}$.

$-\mathrm{val}\, c_\alpha + \alpha w$ is called the <u>w-weight</u> of the term $c_\alpha \underline{x}^\alpha$. The initial form is thus the sum of the classes of the terms of biggest w-weight.

<u>Example:</u>

$$f = (t+t^2)x + 2t^2 y + 3t^4 z$$

$$\in \mathbb{C}\{\{t\}\}[x,y,z]$$

$w = (0,0,0)$ $\qquad W = \max\{-1,-2,-4\} = -1$

$$\overline{t^W f(t^{-w_1}x, t^{-w_2}y, t^{-w_3}z)} = \overline{t^{-1}\left((t+t^2)x + 2t^2y + 3t^4z\right)}$$

$$= \overline{(1+t)x + 2ty + 3t^3z} = x = \mathrm{in}_w f$$

$w = (-4, -2, 0)$

$W = \max\{-1-4,\ -2-2,\ -4\} = -4$

$$\text{in}_w f = \overline{t^{-4}(t+t^2)\, t^4 x + t^{-4}\, 2t^2 t^2 y}$$

$$\overline{+\ t^{-4}\, 3t^4 t^0 z} =$$

$$\overline{(t+t^2)x + 2y + 3z} = 2y + 3z.$$

<u>Proof of</u> Kapranov's theorem 2.5, Part II:

„$\supset$" We do induction on n.

The induction beginning $n=1$ asks us to construct a 0 of a Puiseux series polynomial of a given valuation. This can be done with the algorithm of Thm 2.1 (Puiseux series are algebraically closed).

$n-1 \to n$: Let $w \in V(\text{Trop} f) \cap \mathbb{Q}^n$

We want to lift w to a Puiseux series x (with $-\text{val}\, x = w$) s.th. $f(x) = 0$.

As $w \in V(\text{Trop} f) \Rightarrow$ the max $\text{Trop} f$ is attained at least twice

$\Rightarrow$ $in_w f$ has at least two terms

$\overset{2.6}{\Rightarrow}$ $\exists$ zero $c = (c_1, \ldots, c_n)$ of $in_w f \in (\mathbb{C}^*)^n$.

Case 1: $\exists j$: $in_w f(x_1, \ldots, c_j, \ldots) \neq 0$

OE $j = 1$. Set $w = (w_1, w')$, $x = (x_1, x')$, $c = (c_1, c')$ and consider

$$\tilde{f}(x') := f(c_1 t^{-w_1}, x')$$

Then $\tilde{f}(t^{-w_2} x_2, \ldots, t^{-w_n} x_n) =$

$f(c_1 t^{-w_1}, t^{-w_2} x_2, \ldots, t^{-w_n} x_n) =$

$$\underbrace{in_w f(c_1, x')}_{\neq 0} \cdot t^{-\operatorname{Trop} f(w)} + \text{higher order terms}$$

$\Rightarrow$ $\operatorname{Trop} \tilde{f}(w') = \operatorname{Trop} f(w)$ and

$in_{w'} \tilde{f}(x') = in_w f(c_1, x')$

$\Rightarrow$ $in_{w'} \tilde{f}(c') = 0$, and as $c' \in (\mathbb{C}^*)^{n-1}$

$\overset{2.6}{\Rightarrow}$ $in_{w'} \tilde{f}$ has at least two terms

$\Rightarrow$ $w' \in V(\operatorname{Trop}(\tilde{f}))$

By induction assumption, we can lift w' to x' and add $c_1 t^{-w_1}$

as first component.

<u>Case 2:</u> Assume $in_w f(x_1, \dots, c_j, \dots, x_n) = 0 \ \forall j$

Write

$$in_w f = (x_1 - c_1)^k (x_2 - c_2) \cdots (x_n - c_n) \cdot g(x_1, \dots, x_n)$$

with $g(c_1, x') \neq 0$

Let $\tilde{f}(x') := f((c_1 + t^{\frac{1}{k}}) t^{-w_1}, x')$

Then $\tilde{f}(t^{-w_2} x_2, \dots, t^{-w_n} x_n) =$

$f((c_1 + t^{\frac{1}{k}}) t^{-w_1}, t^{-w_2} x_2, \dots, t^{-w_n} x_n) =$

$in_w f(c_1 + t^{\frac{1}{k}}, x_2, \dots, x_n) \cdot t^{-Trop(f)(w)} + h.o.t. =$

$t^{Trop(f)(w)} (t^{\frac{1}{k}})^k (x_2 - c_2) \cdots (x_n - c_n)\, g(c_1 + t^{\frac{1}{k}}, x_2, \dots, x_n) +$ h.o.t.

As $g \in \mathbb{C}[x_1, \dots, x_n]$, $g(c_1 + t^{\frac{1}{k}}, x_2, \dots, x_n)$

$= g(c_1, x_2, \dots, x_n) +$ terms of order at least $\frac{1}{k}$

$\Rightarrow Trop \tilde{f}(w') = Trop f(w) + 1$,

$in_{w'} \tilde{f} = (x_2 - c_2) \cdots (x_n - c_n) \cdot g(c_1, x_2, \dots, x_n)$

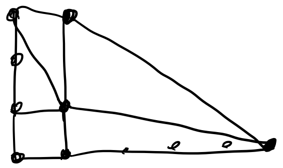
$\Rightarrow in_{w'} \tilde{f}(c') = 0$, $w' \in V(Trop \tilde{f})$

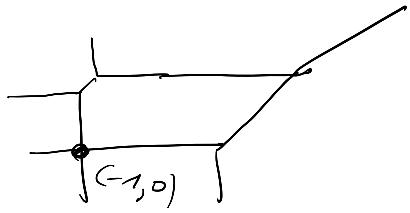
and we can use induction again. $\square$

Example:

$$f = -3t^2 + 3tx - t^2 y + txy - t^3 xy^4 + (t^4+t^5)y^4 + x^5$$

$$\operatorname{Trop} f = \max\{-2, -1+x, -2+y, -1+x+y, -3+x+4y, -4+4y, 5x\}$$





$w = (-1,0) \in V(\operatorname{Trop} f)$.

$\operatorname{in}_w f = -3 + 3x - y + xy \qquad \operatorname{in}_w f(1,-3) = 0$

As $\operatorname{in}_w f(1,y) = \operatorname{in}_w f(x,-3) = 0$, we are in Case 2.

Replace $x = t + t^2$ in f, write $\operatorname{in}_w f = (x-1)\underbrace{(y+3)}_{g}$ and

$$\tilde{f}(y) = f(t+t^2, y) =$$

$$\underline{-3t^2} + \underline{3t(t+t^2)} - \underline{t^2 y} + \underline{t(t+t^2)} y$$

$$- t^3(t+t^2)y^4 + (t^4+t^5)y^4 + (t+t^2)^5 =$$

$$3t^3 + t^5 + 5t^6 + 10t^7 + 10t^8 + 5t^9 + t^{10} + t^3 y$$

$0 \in \operatorname{Trop} \tilde{f}$, here can can solve for y and obtain

$$y = -3 - t^2 - 5t^3 - 10t^4 - 10t^5 - 5t^6 - t^7.$$

So $(x, y) = (t + t^2, \quad " \quad)$

satisfies $(-\mathrm{val}\,x, -\mathrm{val}\,y) = (-1, 0)$

and $f(x, y) = 0$ as required.

3. Abstract varieties

Read Def 3.3, 3.4, Lemma 3.1,

Questions / Activities 3.2, Def 3.5,

Lemma 3.2, Questions / Activities 3.3

in Renzo's notes.

We include a proof of Lemma 3.2

here:

3.1 Lemma

Let A, B, C be finite groups and

$$0 \longrightarrow A \xrightarrow{g} B \xrightarrow{f} C \longrightarrow 0$$ a

short exact sequence.

Then $|A| \cdot |C| = |B|$.

Proof:

For $g, h \in B$ set

$$g \sim h :\Leftrightarrow f(g) = f(h) \in C$$

In each equivalence class, there

are $|\ker f|$ elements $\Rightarrow$

$|B| = |B/\sim| \cdot |\mathrm{Ker} f|$. But $|B/\sim| = |C|$
as f is surjective, and
$\mathrm{Ker} f = \mathrm{Im}\, g \Rightarrow |\mathrm{Ker} f| = |\mathrm{Im}\, g| = |A|$
where the latter equality holds as
g is injective $\Rightarrow |B| = |C| \cdot |A|$.
$\square$

<u>Proof of</u> Renzo's Thm. 3.2:

We have to show that $f_* \Sigma_1$ is balanced.
Let $\tau' \in f_* \Sigma_1$ be of codim 1
and let $\tau \in \Sigma_1$ of codim 1 with $f(\tau) = \tau'$. Around τ we have the balancing condition:

$$\sum_{\tau < \sigma} \omega_{\Sigma_1}(\sigma) \cdot u_{\sigma/\tau} = 0 \text{ in } \mathbb{R}^M / \operatorname{span} \tau$$

Apply f to this equation:

$$\sum_{\tau < \sigma} \omega_{\Sigma_1}(\sigma) \cdot f(u_{\sigma/\tau}) = 0 \text{ in } \mathbb{R}^{M'} / \operatorname{span} \tau'$$

Let $\tau' < \sigma'$ and $\tau < \sigma$ with $f(\sigma) = \sigma'$.

The primitive vector $u_{\mathfrak{z}'/\tau'}$ and $u_{\mathfrak{z}/\tau}$ satisfy:

$$f(u_{\mathfrak{z}/\tau}) = \left| \frac{\operatorname{span}(\mathfrak{z}') \cap \mathbb{Z}^{M'}}{\operatorname{span}(\tau') \cap \mathbb{Z}^{M'} + \mathbb{Z} \cdot f(u_{\mathfrak{z}/\tau})} \right| \cdot u_{\mathfrak{z}'/\tau'}$$

if f is injective on $\mathfrak{z}$ and $f(u_{\mathfrak{z}/\tau}) = 0$ else.

The following sequence is exact:

$$0 \to \frac{\operatorname{span} \tau' \cap \mathbb{Z}^{M'}}{f(\operatorname{span} \tau \cap \mathbb{Z}^M)} \longrightarrow \frac{\operatorname{span} \mathfrak{z}' \cap \mathbb{Z}^{M'}}{f(\operatorname{span} \mathfrak{z} \cap \mathbb{Z}^M)}$$

$$\longrightarrow \frac{\operatorname{span} \mathfrak{z}' \cap \mathbb{Z}^{M'}}{\operatorname{span} \tau' \cap \mathbb{Z}^{M'} + \mathbb{Z} \cdot f(u_{\mathfrak{z}/\tau})} \to 0$$

$$\Rightarrow \left| \frac{\operatorname{span} \mathfrak{z}' \cap \mathbb{Z}^{M'}}{\operatorname{span} \tau' \cap \mathbb{Z}^{M'} + \mathbb{Z} \cdot f(u_{\mathfrak{z}/\tau})} \right|$$

$$= \frac{1}{\left| \frac{\operatorname{span} \tau' \cap \mathbb{Z}^{M'}}{f(\operatorname{span} \tau \cap \mathbb{Z}^M)} \right|} \cdot \left| \frac{\operatorname{span} \mathfrak{z}' \cap \mathbb{Z}^{M'}}{f(\operatorname{span} \mathfrak{z} \cap \mathbb{Z}^M)} \right|$$

Insert this in the above, the first factor is the same for all summands and can be taken out, so we obtain:

$$\sum_{\substack{\tau \subset \sigma \\ f|_\sigma \text{ injective}}} W_{\Sigma_1}(\sigma) \left| \frac{\operatorname{span} \sigma' \cap \mathbb{Z}^{M'}}{f(\operatorname{span} \sigma \cap \mathbb{Z}^M)} \right| \cdot u_{\sigma'/\tau'} = 0$$

in $\mathbb{R}^{M'}/\operatorname{span} \tau'$

We now sum over all τ with $f(\tau) = \tau'$:

$0 =$

$$\sum_{\substack{\tau \\ f(\tau)=\tau'}} \sum_{\substack{\tau \subset \sigma \\ f|_\sigma \text{ inj.}}} W_{\Sigma_1}(\sigma) \left| \frac{\operatorname{span} \sigma' \cap \mathbb{Z}^{M'}}{f(\operatorname{span} \sigma \cap \mathbb{Z}^M)} \right| \cdot u_{\sigma'/\tau'} =$$

$$\sum_{\tau' \subset \sigma'} \left(\sum_{\substack{\sigma \\ f(\sigma)=\sigma'}} W_{\Sigma_1}(\sigma) \left| \frac{\operatorname{span} \sigma' \cap \mathbb{Z}^{M'}}{f(\operatorname{span} \sigma \cap \mathbb{Z}^M)} \right| \right) \cdot u_{\sigma'/\tau'} =$$

$$\sum_{\tau' \subset \sigma'} W_{f_* \Sigma_1}(\sigma') \cdot u_{\sigma'/\tau'} \quad \text{in } \mathbb{R}^{M'}/\operatorname{span} \tau'$$

thus the balancing condition is satisfied. $\square$

Read Def. 3.6, Example 3.2 and Lemma 3.3 in Renzo's notes.

Proof of Lemma 3.3 in Renzo's notes:

As Σ_1 is irred, $\mathrm{supp}(\Sigma_2) = \mathrm{supp}(\Sigma_1)$

Replace Σ_1 and Σ_2 with refinements, s.th. both fans have the same cones, and only the weights are potentially different. Set $\lambda := \min_{\substack{\sigma \in \Sigma_1 \text{ of} \\ \text{top dim}}} \left\{ \frac{w_{\Sigma_2}(\sigma)}{w_{\Sigma_1}(\sigma)} \right\}$

and choose $\alpha \in \mathbb{Z}_{>0}$ with $\alpha\lambda \in \mathbb{Z}_{>0}$.

Take a new weight function

$$w(\sigma) = \alpha\left(w_{\Sigma_2}(\sigma) - \lambda\, w_{\Sigma_1}(\sigma)\right)$$

Then $w(\sigma) \in \mathbb{Z}$, and $w(\sigma) = 0$ at least once.

Take the fan $\Sigma = \{\sigma \in \Sigma_1$ top-dim with $w(\sigma) > 0$ and their faces$\}$.

Then Σ is balanced, as Σ_1 and Σ_2 are, but if $\Sigma \neq \emptyset$ then $\mathrm{supp}(\Sigma) \subsetneqq \Sigma_1$ $\lightning$ to Σ_1 irred

$\Rightarrow \Sigma = \emptyset \Rightarrow w_{\Sigma_2}(\sigma) = \lambda\, w_{\Sigma_1}(\sigma)$

$\forall\, \sigma \in \Sigma_1$ top-dim. $\square$

Read Def 3.7, Thm. 3.1 and
Remark 3.1 in Renzo's notes.

Sections 4, 5 and 6:
Read in Renzo's notes.