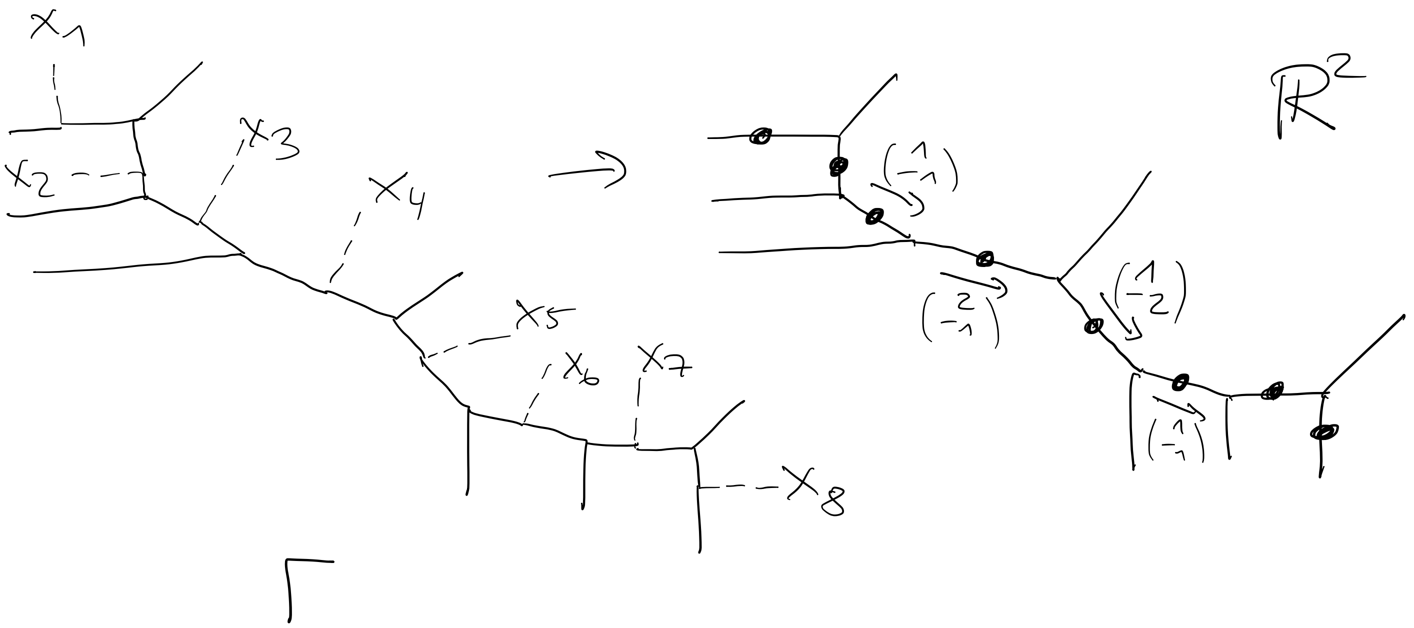
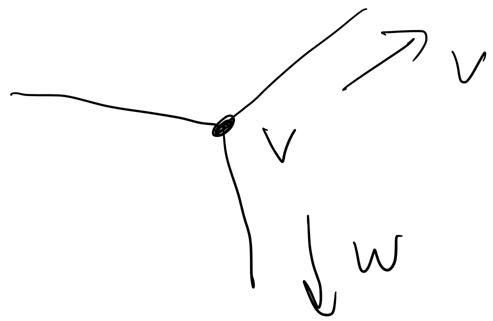
1. Exercises:

1. Recall the definition of N_d^{trop}.
   With what multiplicity does a tropical stable map (Γ, φ) count towards N_d^{trop}?
2. Recall the statement of the Correspondence Theorem.
3. Compute the multiplicity of the following (Γ, φ):



Def:

Let V be a 3-valent vertex of a plane tropical curve C.



Define $\operatorname{mult}_C(V) = |\det(v, w)|$.

We say C is simple, if its dual Newton subdivision contains only triangles and parallelograms.

For a simple C, we define

$$\operatorname{mult}(C) = \prod_{V \text{ 3-valent}} \operatorname{mult}_C(V).$$

4. Formulate a conjecture relating $\operatorname{mult}(C)$ with the multiplicity from before.

5. Prove the conjecture.

6. Express the multiplicity of a plane tropical curve in terms of its dual Newton subdivision.

2.+3. Counting plane tropical curves of higher genus

<u>Def</u> An (explicit) abstract tropical curve is a metric graph Γ with unbounded edges called ends of infinite length.

The <u>genus</u> of a connected tropical curve is the first Betti number of the underlying graph.

Due to the Euler characteristics, the genus equals

$$g = 1 - \#\text{Vertices} + \#\text{bounded edges}$$

If Γ is not connected, but consists of connected components $\Gamma_1, \ldots, \Gamma_r$ of genus $g(\Gamma_i) = g_i$, then the genus of Γ is defined to be

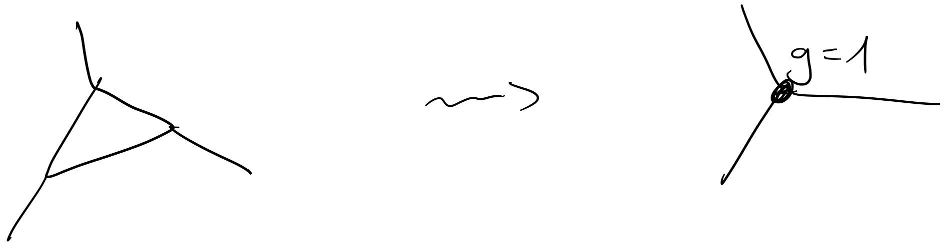
$$g(\Gamma) := \sum_{i=1}^{r} g_i - r + 1$$

Def A <u>parametrized tropical plane curve</u> is a tuple (Γ, φ) s.th. Γ is an abstract tropical curve and φ is a map which is integer affine linear locally on each edge s.th. the balancing condition is satisfied at every vertex.

The genus of (Γ, φ) is the genus of Γ.

The degree of (Γ, φ) is the multiset of directions of its uncontracted ends. (As usual, we call it <u>degree d</u> if the multiset consists of $d \cdot \binom{-1}{0}$, $d \cdot \binom{0}{-1}$ and $d \cdot \binom{1}{1}$.)

<u>Note:</u> We do not need to label the uncontracted ends, since we do not need to construct moduli spaces. In fact, the construction of moduli spaces poses new and interesting challenges, since we need to include limits for families of curves for which „genus gets lost", e.g.:



In general, this means we have to allow abstract tropical curves with „local genus hidden at vertices".
For the purpose of this class, this is not important.

<u>Def</u> A parametrized tropical plane curve is called <u>simple</u> if Γ is 3-valent and the Newton subdivision dual to $\varphi(\Gamma) \subset \mathbb{R}^2$ contains only triangles and parallelograms.

<u>Def</u> Fix $n = 3d+g-1$ points $p_1, \dots, p_n$ in $\mathbb{R}^2$ in general position.

Consider parametrized tropical plane curves of degree d and with n marked contracted ends $x_1, \dots, x_n$.

Let S be the set of such curves satisfying $\varphi(x_i) = p_i$.

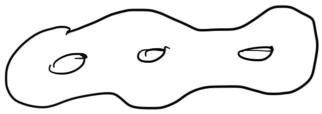
Define

$$N_{d,g}^{trop} := \sum_{(\Gamma,\varphi)\in S} \operatorname{mult}(\Gamma,\varphi).$$

Here, we do not require the Γ to be connected.

If we do that, the number we receive in that way is $N_{d,g}^{trop, irr}$ (for irreducible).

As all algebraic plane curves over the complex numbers are Riemann surfaces, they have a well-defined genus.

.

Note: curves of genus 0  are rational, i.e. they can be parametrized by a line $\mathbb{P}^1_{\mathbb{C}}$.

Note: If an algebraic curve C is not irreducible but has the irreducible components $C_1, \dots, C_r$ of genus $g(C_i) = g_i$, then the genus of C is defined to be

$$g(C) = \sum_{i=1}^{r} g_i - r + 1.$$

We can define the numbers $N_{d,g}$ and $N_{d,g}^{irr}$ in algebraic geometry analoguously.

We then have

<u>Correspondence Thm</u> (Mikhalkin)

$$N_{d,g} = N_{d,g}^{trop}$$

$$N_{d,g}^{irr} = N_{d,g}^{trop, irr}$$

Exercises / Discussions:

1. Construct examples of abstract tropical curves of genus 1.

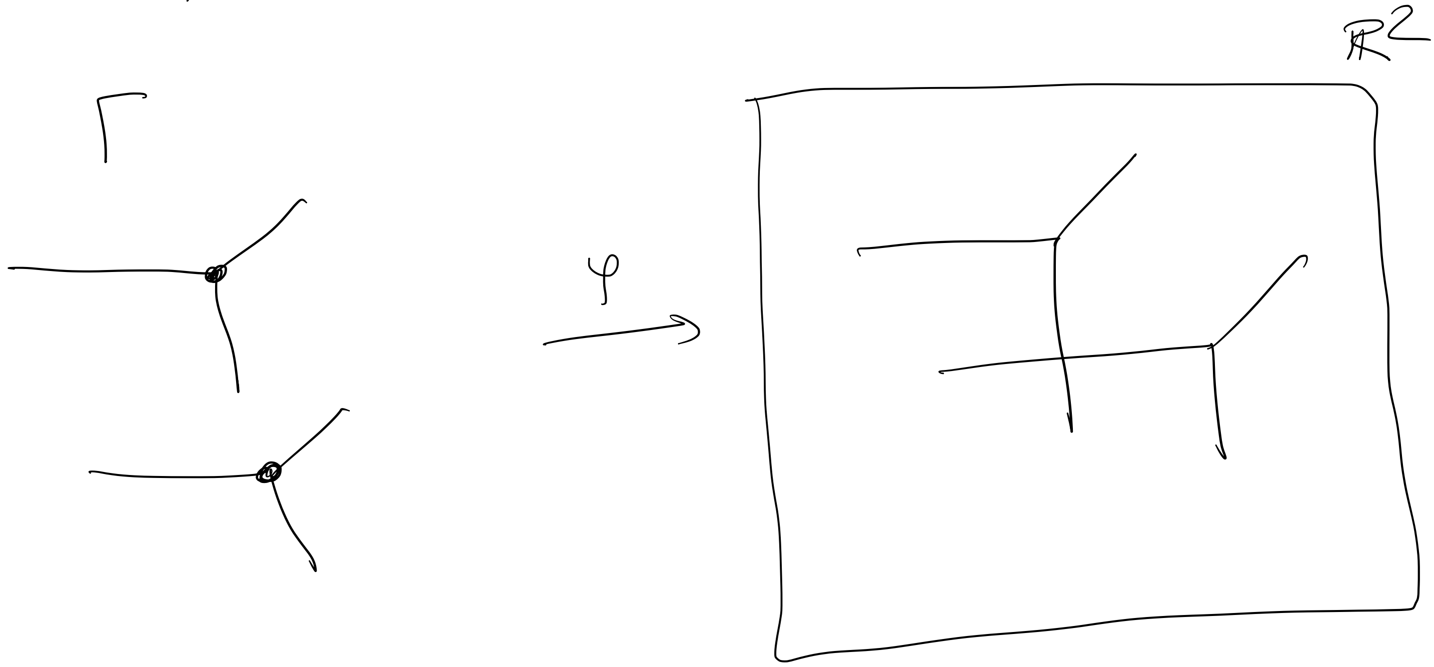
2. Construct an example of a simple parametrized tropical plane curve of degree 3 and genus 1.

3. Find a parametrized tropical plane curve of degree 3 and of multiplicity 1, 3 and 4. Is there one of multiplicity 2?

4. How can we describe the set of all parametrized tropical plane curves of a fixed combinatorial type?

5. Algebraic geometry:
What is the genus of a reducible conic? What irreducible components can we have?
How many reducible conics are there through 4 points in general position?

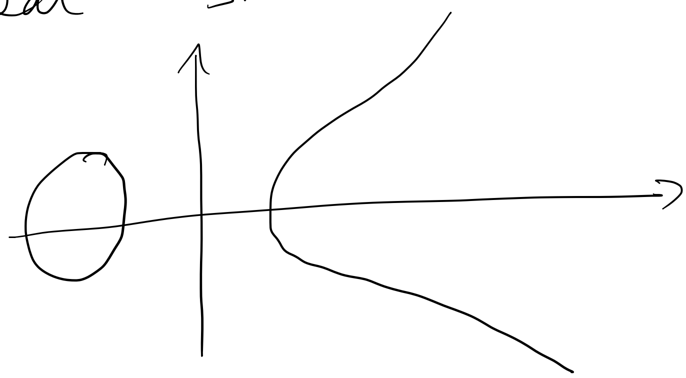
6. What is the genus of (Γ, φ)?



7. Compute $N^{trop}_{2,-1}$.

8. Algebraic geometry:
Plane conics are rational (as an example, think of $V(y-x^2) \subset \mathbb{C}^2 \subset \mathbb{P}^2_{\mathbb{C}}$)

Smooth plane cubics have genus 1.
Real sketch:



9. The subset of points $(p_1, \dots, p_n) \in \mathbb{R}^{2n}$ s. th. the set S of all parametrized tropical curves with $\varphi(x_i) = p_i$ contains only simple curves is dense in the Euclidean topology.

10. Determine $N_{2,0}^{trop}$.

11. Let (Γ, φ) be a simple parametrized tropical plane curve. The genus equals the number of interior lattice points that appear in the dual Newton subdivision minus the number of parallelograms that appear in the dual Newton subdivision.

12. Algebraic geometry: Determine $N_{2,0}$ and $N_{3,1}$.
Hint: A plane conic is given by a polynomial of the form $a x^2 + b xy + c y^2 + d x + e y + f$ (dehomogenized).

The possible coefficients $(a, \ldots, f)$ can be viewed as a point in $\mathbb{P}^5$.

Fix a point p_1 in the plane $\mathbb{P}^2_{\mathbb{C}}$.

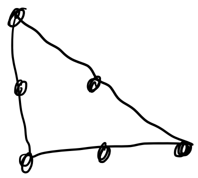
Which subset in $\mathbb{P}^5$ parametrizes conics that pass through p_1?

4. + 5. Lattice paths

Read definition 2.12, 2.13, 2.14 in arXiv: 0504392.

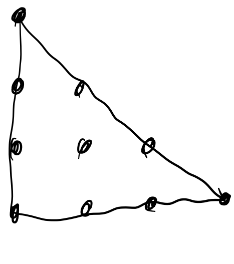
Exercises / Discussions:

1. Count all lattice paths in



with 5, and with 4 steps, with multiplicity.

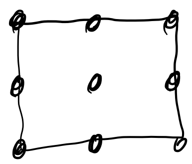
2. Count all lattice paths in



with 9, and with 8 steps, with multiplicity.

3. Discuss generalizing the definition to other polygons.

4. Count all lattice paths in



with 8 steps, with multiplicity.

5. Determine $N^{path}_{d,g}$ for

$$g = \binom{d-1}{2} \quad \text{and} \quad g = \binom{d-1}{2} - 1.$$

6. Draw all Newton subdivisions that one obtains by computing the multiplicity of a path contributing to $N^{path}_{3,0}$, marking the original path with thick lines.
Sketch dual tropical curves.
Is there a configuration of points through which they all fit?

Def

Let L be a line orthogonal to $\{x - \varepsilon y = 0\}$.

Let $p_1, \dots, p_n$ be points on L, such that the distances grow, i.e. $|p_{i+2} - p_{i+1}| \gg |p_{i+1} - p_i|$.

We call such points in Mikhalkin position.

Exercises / Discussions:

7. There are points in general position (for the counting problem $N_{d,g}^{trop}$) in Mikhalkin position.

8. Draw 4 points in Mikhalkin position, and draw the tropical conics of genus -1 through

them.
Compare with Ex 1.
Draw 5 pts and all tropical conics of genus 0, compare with Ex 1.

9. Draw 8 points and rational cubics, compare with Ex 2 and 6.

10. Consider a strip with bounds the line L and a parallel shift.
For the tropical curves you have drawn in 9, what parts in the dual Newton subdivision correspond to the parts in the strip?

11. Given a tropical curve passing through points in

Mikhalkin position, mark the edges in the dual Newton subdivision dual to edges that contain the marked points (contracted edges). What you obtain is a λ-increasing lattice path.

12. Prove:

$$N_{d,g}^{trop} = N_{d,g}^{path}$$

13. How can irreducibility be detected in the lattice path setting?

6. The Caporaso - Harris formula for lattice paths

Read Def 2.1, 2.2, 2.3

and chapter 3 of

arXiv: 0504392

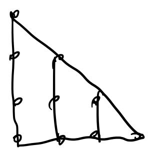
7. The Caporaso - Harris formula for plane tropical curves

Read chapter 4 of

arXiv: 0504392

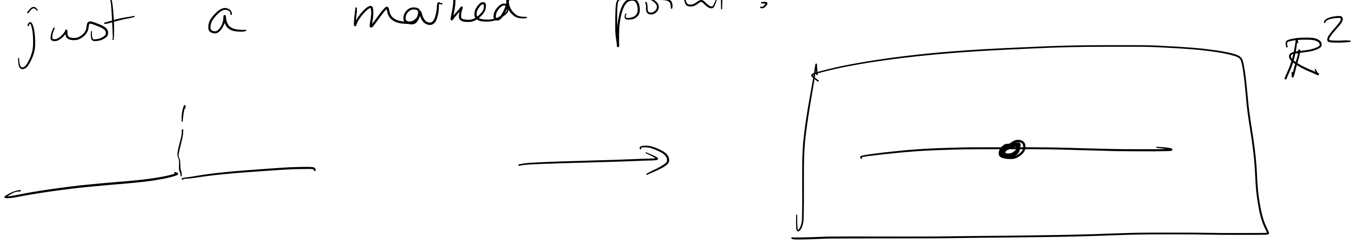
8. Floor diagrams

"Caporaso - Harris ultimately"

Put point conditions in a small horizontal strip.
(Careful, observe the difference to Mikhalkin position.)
Every plane tropical curve passing through such points is
floor decomposed (i.e. its dual Newton subdivision contains all vertical line segments: )

Def: A horizontal edge is called an elevator, a connected component of a tropical curve minus its elevators is called a floor.
Note that a floor may consist of

just a marked point:



Exercise/Discussions:

1. Every floor contains precisely one marked point.

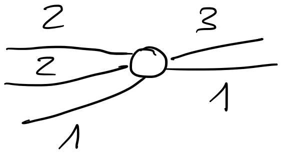
2. Draw a picture of a floor decomposed curve.
   Shrink the floors to vertices.
   Color the vertex black if the floor consists of just a marked point and white otherwise.

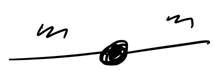
   Discuss the features of the graph you obtain.

Def

A <u>floor diagram</u> is a graph on a linearly ordered vertex set, with vertices colored black and white and edges equipped with weights, s. th.

- d unbounded edges of weight 1 each point to the left
- each white vertex is of <u>divergence 1</u>, i.e. the sum of the weight of the incoming edges (from the left) minus the outgoing edges (to the right) equals 1.



- each black vertex is of valence 2 and divergence 0: 

- the graph is bipartite.

Def We define $N_{d,g}^{floor}$ to be the weighted count of floor diagrams of genus g (with d weight 1 left ends) on $3d+g-1$ vertices, each counted with multiplicity equal to the product of its edge weights.

Exercises / Discussions:

3. Compute $N_{3,0}^{floor}$

4. Prove $N_{d,g}^{floor} = N_{d,g}^{trop}$

5. Can the definition be generalized
   - to other polygons?
   - to count the numbers showing up in the Caporaso-Harris formula?

9. + 10. + 11. Hurwitz numbers in algebraic geometry resp. topology

Cavalieri, Miles:
Riemann surfaces and algebraic curves: a first course in Hurwitz theory

9. Read 4.1, 4.2, 4.3, 4.4
10. Read 6.1, 6.2, 7.1, 7.2
11. Read 7.3, 7.4

We will focus on <u>double Hurwitz numbers of $\mathbb{P}^1$</u>.

Let $\mu, \nu \vdash d$ be two ramification profiles.

We let

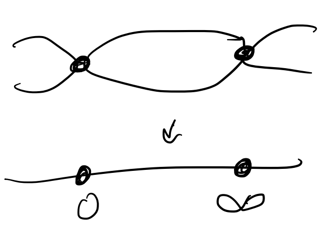
$$H_g(\mu, \nu) := \sum_{f \in S} \frac{1}{\mathrm{Aut}(f)},$$ where

$S =$

$\{ f : C \to \mathbb{P}^1$, s.th.

- C is of genus g,
- f has ramification profile μ over 0
- f has ramification profile ν over ∞,
- f has simple ramification over $n = 2g-2+\ell(\mu)+\ell(\nu)$ other fixed points in $\mathbb{P}^1 \}$

<u>Ex</u>: $H_0(2,2) = 1/2$



$$\begin{array}{c} x \\ \downarrow \\ x^2 \end{array}$$

12. Tropical double Hurwitz numbers

<u>Def:</u> A tropical cover of $\mathbb{R}$ is a tuple (Γ, φ) s. th.

- Γ is an abstract tropical curve
- φ is integer affine linear on each edge, i.e. of the form $[0, \ell(e)] \ni t \mapsto a + w \cdot t \in \mathbb{R}$, where $a \in \mathbb{R}$ and $w \in \mathbb{Z}$. The number $|w|$ is called the <u>weight</u> of the edge e.
- the balancing condition is satisfied at every vertex, i.e. the sum of the incoming weights equals the sum of the outgoing weights.

The *genus* of a tropical cover is the genus of Γ.

The ramification profile (over $-\infty$) is the multiset of weights of the left ends, the profile (over ∞) the multiset of weights of right ends.

The *degree* of a tropical cover is the weighted number of preimages of a generic point in $\mathbb{R}$.

A simple ramification point is a vertex of Γ.

Exercises / Discussions:

1. Construct an example of a tropical cover
2. The degree is well-defined.

3. Use the Euler characteristics for graphs to determine the number of vertices of a tropical cover (Γ, φ) for fixed ramification profiles (μ, ν) and genus.

4. Discuss a possible definition of isomorphism and automorphism of a tropical cover.

5. Express the size of the automorphism group of a tropical cover in terms of the combinatorics of Γ.

6. Define the combinatorial type of a tropical cover in analogy to the case of plane tropical curves.

7. What parametrizes all tropical covers of a given combinatorial type?

We glue the answers for 7 in an appropriate way to obtain a moduli space $M_g(\mu,\nu)$ parametrizing tropical covers of genus g and with ramification profiles μ (resp. ν) over $-\infty$ (resp. ∞).

Def:
Let $br: M_g(\mu,\nu) \longrightarrow \mathbb{R}^n$ (resp. $\mathrm{Sym}_n(\mathbb{R})$) be the branch morphism that evaluates the images of the vertices.

Exercises / Discussions:

8. The branch morphism is

locally a linear map.

9. (In the rational case:)
What is the multiplicity of the branch morphism at (Γ, φ)?
Express it in terms of the combinatorics of (Γ, φ).

<u>Def</u>

$$H_g^{trop}(\mu, \nu) := \deg(br).$$

<u>Exercises / Discussions</u>:

10. Compute $H_0^{trop}((3,1),(2,2))$

13. The Correspondence Theorem for Hurwitz numbers

The goal of this section is to prove the

Correspondence Theorem

$$H_g(\mu,\nu) = H_g^{trop}(\mu,\nu)$$

This is important, because as soon as we know this, we can study double Hurwitz numbers in algebraic geometry merely by counting appropriate graphs with multiplicity!

This helped e.g. to obtain new results about the structure of double Hurwitz numbers.

Recall that by Cavalieri-Miles, Theorem 7.3.1

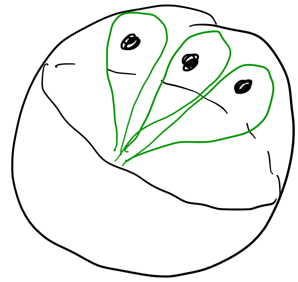
$$H_g(\mu,\nu) = \frac{1}{d!}\#\left\{ \begin{array}{l} \text{Monodromy representations} \\ \pi_1(\mathbb{P}^1_{\mathbb{C}} \setminus \{0,\infty,p_1,\dots,p_n\}) \\ \longrightarrow S_d \quad \text{of type} \\ (g,\ \mu,\ \nu,\ \text{simple} \dots) \end{array} \right\}$$

$$\pi_1(\mathbb{P}^1_{\mathbb{C}} \setminus \{0,\infty,p_1,\dots,p_n\}) = \mathbb{F}_{n+1}$$

(the free group in $n+1$ generators)

$$\cong \mathbb{F}_{n+2} / \langle l_1 \cdots l_{n+2} \rangle$$

where $l_i \hat{=}$ loop around p_i



A monodromy representation thus consists of a tuple $(b_\nu, \tau_n, \dots, \tau_1, b_\mu)$ s.th.

- b_ν is of type ν, b_μ of type μ
- τ_i are of simple type, i.e. τ_i is a transposition, and
- $b_\nu \circ \tau_n \circ \dots \circ \tau_1 \circ b_\mu = id$

$$\Longleftrightarrow \quad \tau_n \circ \dots \circ \tau_1 \circ b_\mu = b_\nu^{-1},$$

where b_ν^{-1} is also of type ν.

Exercises / Discussions:

1. If c is a cycle of length k and τ_i is a transposition, what can $\tau_i \circ c$ be?

2. What can $\tau_i \circ \sigma_\mu$ be?

3. Find a monodromy representation that counts towards $H_0((3,1),(2,2))$

4. Given a monodromy representation $(\tau_n, \dots, \tau_1, \sigma_\mu)$, sketch the type of the permutations
σ_μ, $\tau_1 \circ \sigma_\mu$, $\tau_2 \circ \tau_1 \circ \sigma_\mu$, ...
Sketch this for your representation from 3. What picture do you obtain?

5. Given one picture as in 4., how many monodromy representations would produce it? Identify the following contributions:

- the number n_μ of permutations in S_d of type μ
- $|\mathrm{Aut}(\mu)|$
- for every vertex, the product of the weights of the incoming edges
- for every balanced fork or wiener, a factor of $1/2$.

6. Let $\mu = (\mu_1, \dots, \mu_r)$. Show:

$$n_\mu \cdot |\mathrm{Aut}(\mu)| \cdot \prod_{i=1}^{r} \mu_i = n!$$

7. Prove the correspondence theorem.