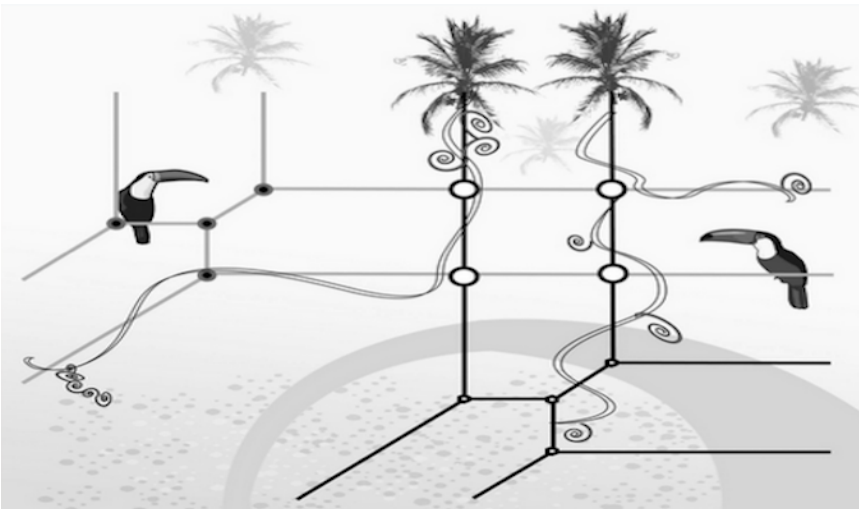
Counting Tropical Plane Curves

Renzo Cavalieri
following a master class by Hannah Markwig at Stockholm University

Notes for Students



Mini-course at Universidad de Costa Rica

San Jose', August 5 - 13, 2019

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Introduction

The main goal of this minicourse is to introduce students to tropical geometry and (some of) its enumerative applications. Tropical geometry is a combinatorialization of ordinary geometry. By either doing algebraic geometry over the *tropical semi-field* or over *fields with a valuation*, one may associate piecewise linear objects to ordinary algebraic varieties. Remarkably such highly degenerate objects retain a lot of geometric information about the original algebraic varieties. For example, the notion of dimension is preserved: thus to an algebraic curve is associated a graph. Further, the genus of the curve generically is equal to the genus of the graph. In the last twenty-some years many groups of researchers have explored these connections, providing a wealth of *correspondence theorems*: statements on how to recover classical information from its tropical counterpart. In this minicourse we will focus in particular on a celebrated result of Mikhalkin, which shows that the count of rational tropical plane curves of degree d through $3d-1$ points in general position equals the classical count. The material will be distributed among the six classes as follows:

Day 1: The tropical semi-field. Tropical plane curves as determined by polynomial equations valued in the tropical semi-field. The Newton polygon of a polynomial in two variables. Tropical curves are dual to a subdivision of a Newton polygon. The notion of balancing.

Day 2: The field of Puiseaux series. Axiomatic definition of a field with a valuation. Algebraic curves over valued fields and their tropicalization. Intersection multiplicity of two tropical curves. Bézout's theorem for plane tropical curves.

Day 3: Cones, fans, and their morphisms. The notion of a balanced fan, and criteria to verify that a fan is balanced. The push-forward of a balanced fan is balanced.

Day 4: Moduli spaces of tropical rational curves $M_{0,n}^{trop}$ as balanced fans. The forgetful morphisms.

Day 5: Moduli spaces of tropical rational stable maps to the tropical plane. Evaluation morphisms and incidence conditions. The key enumerative question (*how many plane curves of a given degree pass through a number of points in general position?*), and how to interpret it in terms of a question of intersection on a moduli space of tropical stable maps.

Day 6: Mikhalkin's correspondence theorem, recovering Kontsevich's recursive formula to answer the key enumerative question.

The material will be covered partly through lectures, partly through direct involvement of the students, whom will be asked to work on exercises and to complete worksheets aimed at helping them assimilate the material.

Acknowledgements

These notes are heavily based on a mini-course taught by Hannah Markwig during a master class at Stockholm University in the summer of 2017. My main contribution consisted in selecting some of the material to make it accessible to undergraduate students in the timeframe available, and in converting some of the lecture material into exercises and activities. Most of the credit for the choice and organization of the material goes to Hannah, and all of the blame for eventual mistakes rests on me.

References and Resources

There are many introductory resources to tropical geometry. Here is just a very incomplete selection for students that may be interested in reading more.

(1) *Introduction to Tropical Geometry.* Bernd Sturmfels and Diane Maclagan. Graduate Studies in Mathematics, AMS. Freely available online.
(2) *Brief introduction to tropical geometry.* Erwan Brugallè, Ilia Itenberg, Grigory Mikhalkin, Kristin Shaw. arXiv:1502.05950.
(3) *Enumerative tropical algebraic geometry in* $\mathbb{R}^2$ G. Mikhalkin. J. Amer. Math. Soc., 18(2):313 - 377, 2005.
(4) *Moduli spaces of rational tropical curves.* G. Mikhalkin. Proceedings of Gokova Geometry/Topology conference 2006, pages 39 - 51, 2007.
(5) *Fock Spaces and Tropical Curve Counting.* Renzo Cavalieri. https://www.math.colostate.edu//~renzo/CF.pdf

DAY 1

Tropical algebra and geometry

Today we are getting acquainted with tropical algebra, and tropical geometry. Tropical algebra replaces addition and multiplication with two new operations, tropical addition and tropical multiplication. Tropical polynomials then correspond to piecewise linear, continuous functions in ordinary algebra. Given a tropical polynomial in two variables, we can associate to it a tropical plane curve, which is a "stick figure" in the plane. We study the relationship between the tropical polynomial and the shape of the corresponding tropical plane curve, in a similar way to how in analytic geometry we study geometric properties of the curves obtained as the solution sets of their polynomial equations.

DEFINITION 1.1. The **tropical semifield** $\mathbb{T} = (\mathbb{R} \cup \{-\infty\}, \oplus, \odot)$ consists of the real numbers union the symbol $-\infty$, with the two operations:

tropical sum:

$$a \oplus b := max(a, b),$$

where we understand that $-\infty$ is considered to be smaller than any real number.

tropical multiplication:

$$a \odot b := a + b.$$

DEFINITION 1.2. A **tropical polynomial** $p(x_1, \dots, x_n)$ is a finite sum of tropical monomials:

$$p(x_1, \dots, x_n) = \bigoplus_{I=(i_1,\dots,i_n)} \alpha_I \odot x_1^{\odot i_1} \odot \dots \odot x_n^{\odot i_n}, \tag{1}$$

where all the $i_j \in \mathbb{Z}^{\geq 0}$, and $x^{\odot i} = \underbrace{x \odot \dots \odot x}_{i\text{–times}}$.

QUESTIONS/ACTIVITIES 1.1.

(1) Show that $\oplus$ has an identity element. What is it? Show that no element other than the identity has an additive inverse.
(2) Show that $\mathbb{T} \setminus \{-\infty\}$ is an abelian group. What is the multiplicative identity? What is the multiplicative inverse of $a \in \mathbb{R}$?
(3) What is $5^{\odot 0}$?
(4) Given the tropical polynomial $0 \oplus x^{\odot 2}$, what is the coefficient of x? What is the coefficient of $x^{\odot 2}$?

(5) Consider the tropical monomial $\mu(x,y) = 3 \odot x^{\odot 2} \odot y$: show that it gives rise to an affine linear function $f_\mu : \mathbb{R}^2 \to \mathbb{R}$.
(6) Are the tropical polynomials x, $1 \odot x$ and $x \oplus 0$ equal or different?
(7) What kind of function corresponds to the polynomial $p(x) = 0 \oplus 1 \odot x \oplus x^{\odot 2}$? Draw its graph.
(8) Show that the two tropical polynomials $p(x) = 0 \oplus x \oplus (1 \odot x^{\odot 2})$ and $q(x) = 0 \oplus (1 \odot x^{\odot 2})$ give rise to the same function.

Let us recap what we have learned so far about tropical algebra:

- *Most numbers do not have an additive inverse, so we cannot in general solve linear equations!*
- *A number to the zero power is zero, and we don't write monomials whose coefficient is* $-\infty$.
- *Tropical monomials correspond to affine linear functions in ordinary algebra. Tropical polynomials correspond to continuous, piecewise-linear functions.*
- *Different polynomials can give rise to the same function, because some of the monomials may be irrelevant (their graph is always below the graph of some other monomial).*

Next we are going to explore what happens when we do geometry with this strange number system.

DEFINITION 1.3. Let p be a tropical polynomial in n-variables and $f_p : \mathbb{R}^n \to \mathbb{R}$ the corresponding piecewise linear function. We define

$$V(p) := \{\mathbf{x} \in \mathbb{R}^n | \text{the maximum value for } f_p \text{ is attained by at least two monomials}\}.$$

In other words, $V(p)$ is the corner locus of the piecewise linear function f_p, i.e. the values of the domain where f_p is not differentiable.

If p is a polynomial in two variables, we call $V(p)$ a **tropical plane curve**. We denote a tropical plane curve by Γ when focusing on its piecewise linear structure rather than on the tropical polynomial from which it arose.

EXAMPLE 1.1. Let $p(x,y) = x \oplus y \oplus 0$. The associated function is $f_p(x,y) = max(x,y,0)$. The corresponding tropical curve, called a **tropical line** because it is associated to a linear polynomial, is depicted in Figure 1.1.

QUESTIONS/ACTIVITIES 1.2.

(1) Draw the tropical line associated to the tropical polyomial $a \odot x \oplus b \odot y \oplus c$ for $a, b, c \in \mathbb{T}$. Analyse also the case when some of the coefficients equal $-\infty$.
(2) Show that there is a unique tropical line through two general points in the plane. What does "general" mean here?
(3) Show that two general lines intersect in exactly one point. What does "general" mean here?
(4) Draw the tropical curve associated to the tropical polyomials:
    - $(x \odot y) \oplus x \oplus y \oplus 0$.
    - $(x \odot y) \oplus x \oplus y \oplus (-1)$.
    - $(x \odot y) \oplus (-1 \odot x) \oplus y \oplus 0$.

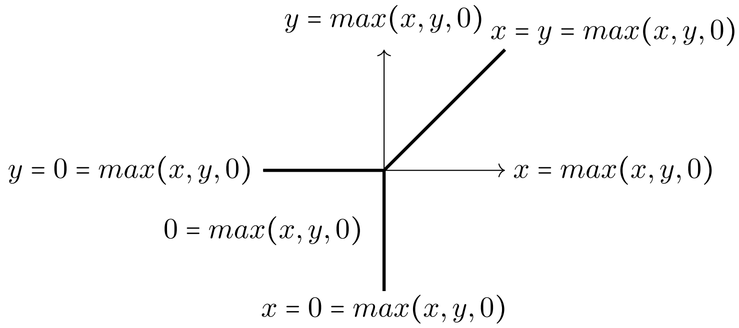


FIGURE 1.1. The tropical line from Example 1.1. The domain plane is subdivided in three parts, according to which of the functions x, y and 0 attains the maximum value. The tropical line $V(p)$ is the tripod separating these regions.

FIGURE 1.2. Examples of the outward normal fan (red) of a polygon (blue).

Given a polygon $P \subset \mathbb{R}^2$, we call the **outward normal fan of** P the collection of half-lines, centered at the origin, which are perpendicular to edges of P, and point outward from P in the sense that they have the same orientation as some parallel half-line which originates inside P, and crosses the edge to go out towards infinity. The examples in Figure 1.2 should help clear any doubt on what this means.

Given a monomial in two variables $x^{\odot m} \odot y^{\odot n}$, one may naturally associate to it the point of the plane (m, n), which always has integer coordinates.

DEFINITION 1.4. Given a polynomial $p(x, y) \in \mathbb{T}[x, y]$, the **Newton polygon** Δ_p of p consists of the convex hull of the points in the plane associated to the monomials with non-$(-\infty)$ coefficients (see Figure 1.3). We call these monomials the **support** of p.

The Newton polygon only cares about which coefficients of p are non-$(-\infty)$. The actual values of the coefficients give us some more refined information. To any monomial $\mu = a_\mu \odot x^{\odot m} y^{\odot n}$ associate the point $P_\mu = (m, n, a_\mu) \in \mathbb{R}^3$. The convex hull of the points P_μ, for all monomials μ in the support of p, is a polytope $\tilde{\Delta}_p \subset \mathbb{R}^3$, that maps onto the Newton polygon via vertical projection. Any face of $\tilde{\Delta}_p$ whose outward normal vector points up (i.e. has positive z coordinate) is called an **upper face**.

DEFINITION 1.5. Given a polynomial $p(x, y)$, the **dual Newton subdivision** of Δ_p is the image via vertical projection of the edges of the upper faces of $\tilde{\Delta}_p$.

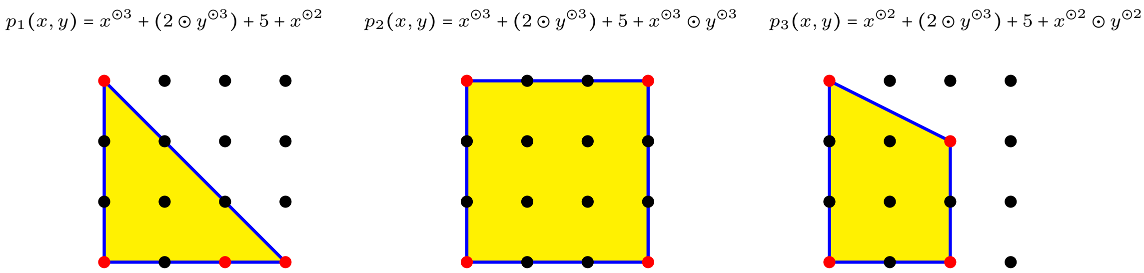
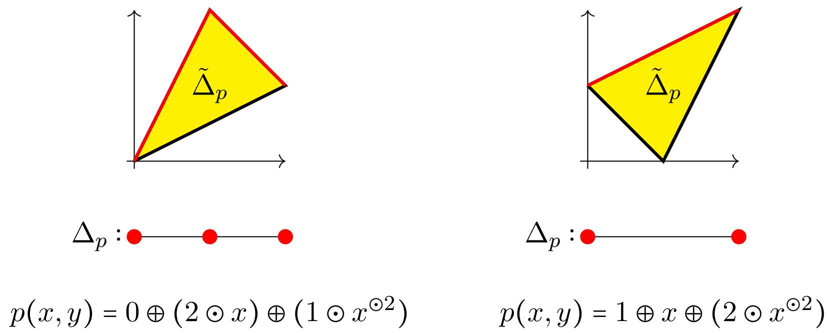


FIGURE 1.3. Three polynomials of degree three and their associated Newton polygons. The red points correspond to the monomials with non-$(-\infty)$ coefficients.

EXAMPLE 1.2. We illustrate these definitions in some one dimensional examples. In the two figures below, the Newton polygon is the segment $[0,2]$ (since the polynomials are univariate and of degree 2). The red dots in the Newton polygons represent the dual Newton subdivisions associated to the two polynomials. The yellow triangles are the $\tilde{\Delta}_p$'s, whose upper edges are colored red.



QUESTIONS/ACTIVITIES 1.3.

(1) Compute the dual Newton subdivisions of the three polynomials from Exercise (4) in the previous activity session. Any relationship to the tropical curves you wrote?
(2) Suppose the coefficients for all monomials in the support of p are equal. What is the dual Newton subdivision of p?
(3) Consider the polynomial $p(x,y) = 0 \oplus a \odot x \oplus x^2$. Describe how the Newton subdivision varies as a varies. Describe the corresponding tropical plane curves.
(4) Associate to two points $P_1 = (m_1, n_1, a)$ and $P_2 = (m_2, n_2, b)$ the affine linear functions (in ordinary algebra) $L_1 = m_1x + n_1y + a$ and $L_2 = m_2x + n_2y + b$. Show that the line where $L_1 = L_2$ is perpendicular to the vertical projection of the segment $\overline{P_1P_2}$.

In these exercises we are starting to see some relationship between the Newton subdivision associated to a polynomial p and the tropical curve of p. The next theorem makes this connection precise.

THEOREM 1.1. *Let $p(x, y)$ be a tropical polynomial. Then the tropical curve $V(p)$ associated to p is dual to the Newton subdivision of p in the following sense:*

(1) *There is a vertex for $V(p)$ for every polygon in the Newton subdivision.*
(2) *There is an edge for $V(p)$ for every edge in the Newton subdivision.*
(3) *The edge joining two vertices of $V(p)$ is perpendicular to the common edge for the corresponding polygons in the Newton subdivision.*

Further, the coordinates of a vertex w of $V(p)$ may be determined by solving the linear system $L_{v_1} = \ldots = L_{v_k}$, where the v_i's are the vertices of the polygon dual to w and the L_{v_i} are the associated affine linear functions as in Exercise (4) *in the last activity session.*

EXAMPLE 1.3. Figure 1 shows three examples of tropical curves dual to Newton subdivisions of a triangle of sidelength 2.

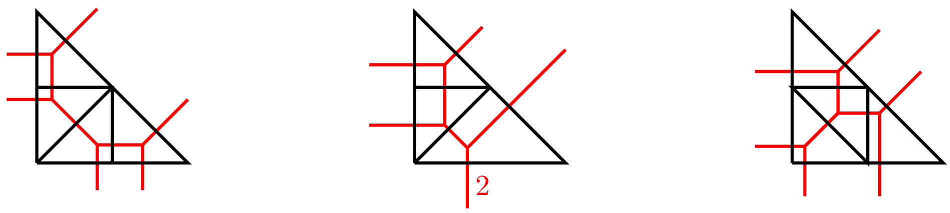


FIGURE 1.4. Examples of tropical conics (red), and their dual Newton subdivisions (black).

For the sake of time, we will not give a complete proof of this theorem. We mention however that the proof follows from carefully combining these ideas:

- The affine linear functions associated to the vertices of $\tilde{\Delta}_p$ are the functions that arise when writing f_p in terms of ordinary algebra.
- For any pair of vertices of $\tilde{\Delta}_p$, consider the segment joining them. We saw in Activity 1.3.4 that the locus of points where the corresponding linear functions in f_p are equal is perpendicular to this segment. A non-empty part of this locus of points appears in the tropical curve $V(p)$ if and only if for some (x, y) the value of those affine linear functions is the maximum of among all affine linear functions in f_p.
- This happens precisely when the segment in question is the edge of an upper face for $\tilde{\Delta}_p$.

Now we add some structure to tropical curves. Let S be a segment in $\mathbb{R}^2$ with endpoints with integer coordinates; if S contains N points with integral coordinates, we say that the **lattice length** of S is equal to $N - 1$.

DEFINITION 1.6. The **weight** of an edge e of a tropical curve is the lattice length of the corresponding edge in the dual Newton subdivision.

We say a vector in $\mathbb{R}^2$ with integer coordinates is **primitive** if the *gcd* of its coordinates is 1.

Definition 1.7. Let Γ be a tropical plane curve. For any oriented edge e of Γ, denote by $\mathbf{p}_e$ the primitive vector of e and w_e the weight of e. For any vertex v of Γ, let $e_1, \dots e_k$ be the edges of Γ adjacent to v, oriented away from the vertex. Then we say Γ is **balanced at** v if

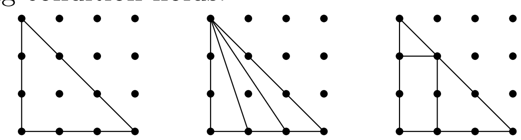
$$\sum_{i=1}^{k} w_{e_i} \mathbf{p}_{e_i} = 0. \tag{2}$$

If a plane tropical curve satisfies the balancing condition at every vertex, we say Γ is **balanced**.

Theorem 1.2. *Plane tropical curves are balanced.*

Questions/Activities 1.4.

(1) Draw the tropical curves which are dual to the following Newton subdivisions, and compute the weights of their edges. Verify that the balancing condition holds.



(2) Prove Theorem 1.2. First reduce it to a statement about lattice polygons (polygons whose vertices have integral coordinates). Then you may do induction on the area of the polygon.

Further exercises

Exercise 1.1. Let $p(x) = a_0 \oplus (a_1 \odot x) \oplus \dots \oplus (a_d \odot x^{\odot d})$ be a tropical polynomial of degree d. Prove that the graph of f_p can have at most d corners. What conditions on the coefficients a_i must hold in order for the graph of f_p to have exactly d corners? In this case, call $x_1, \dots x_d$ the x-coordinate of the corners. Prove that p factors in linear factors:

$$p(x) = a_0 \odot (x \oplus x_1) \odot \dots \odot (x \oplus x_d). \tag{3}$$

We say that the x-coordinates of the corners are the **roots** of the tropical polynomial p.

Exercise 1.2. In analysis we define the exponential function as:

$$e^x := \sum_{n=0}^{\infty} \frac{x^n}{n!}. \tag{4}$$

Let us define the **tropical exponential function** $e^{\odot x}$ by replacing all the operations by their tropical counterparts (be careful: what do division, and factorials correspond to tropically?). Describe the graph of $e^{\odot x}$ and its roots.

Exercise 1.3. Find tropical polynomials whose associated tropical curves are dual to the subdivisions in Example 1.3.

DAY 2

Geometry over a valued field.

Today we introduce the notion of a valued field, through the prototypical example of the field of Puiseaux series. We discover that there is an operation, called tropicalization, that takes an algebraic curve defined over the field of Puiseaux series to a tropical curve. We define multiplicities of intersections of tropical curves, and show that the classical Bzout theorem, which says that two curves of degree d_1 and d_2 intersect in $d_1 d_2$ points, holds in tropical geometry as well.

The field of **Puiseaux series** $\mathbb{C}\{\{t\}\}$ consists of formal expressions of the form

$$c(t) = c_1 t^{a_1} + c_2 t^{a_2} + \dots, \tag{5}$$

where

- c_i are non-zero complex numbers;
- a_i are rational numbers, and $a_i < a_{i+1}$;
- the sum is countable;
- the set of all exponents a_i admits a common denominator.

If we define addition and multiplication of Puiseaux series the same way we do with ordinary power series, we easily see the that set of all Puiseaux series forms a field. In fact it is an algebraically closed field.

The field of Puiseaux series helps us to connect regular and tropical algebraic geometry. In regular algebraic geometry, we like algebraically closed fields, because that guaranteed that polynomials have the amount of roots that they "should have". For example, the second degree equation $x^2 + 1 = 0$ has no solutions over the field of real numbers, but it has two complex solutions, $x = \pm i$. The connection to tropical geometry arises from the fact that one can study the "order of magnitude" (in the sense of calculus) of a Puiseaux series at zero. Before we expand on this connection we introduce the notion of a valuation on a field.

DEFINITION 2.1. A **valuation** on a field K is a function

$$val : K \to \mathbb{R} \cup \{+\infty\}$$

such that:

- $val(x) = +\infty \iff x = 0$;
- $val(xy) = val(x) + val(y)$;
- $val(x + y) \geq min(val(x), val(y))$, with equality holding if $val(x) \neq val(y)$.

QUESTIONS/ACTIVITIES 2.1.

(1) Which of these expressions is a Puiseaux series?
- $c(t) = \frac{1}{t} + t$;
- $c(t) = t^{\frac{1}{3}} + t^{\frac{1}{2}}$;
- $c(t) = t^{\sqrt{2}}$;
- $c(t) = \sum_{i=0}^{+\infty} t^i$;
- $c(t) = \sum_{i=0}^{+\infty} t^{-i}$;
- $c(t) = \sum_{i=-5}^{+\infty} t^i$;
- $c(t) = \sum_{i=1}^{+\infty} t^{\frac{1}{i}}$.

(2) Let K be a field with a valuation. Denote by 1 the multiplicative identity of K. What is $val(1)$? If $x \neq 0$, how does $val(1/x)$ compare to $val(x)$? How about $val(-x)$?

(3) The function

$$val : \mathbb{C}\{\{t\}\} \to \mathbb{R} \cup \{+\infty\} \tag{6}$$

defined by

$$val(0) = +\infty \qquad val(c_1 t^{a_1} + c_2 t^{a_2} + \ldots) = a_1 \tag{7}$$

is a valuation on the field of Puiseaux series. Write down the valuation for all the Puiseaux series from Problem 1. Write an example of two Puiseaux series $c_1(t)$ and $c_2(t)$ such that

$$val(c_1(t) + c_2(t)) > min(val(c_1(t)), val(c_2(t))). \tag{8}$$

Now we explore the relationship between algebraic geometry over a valued field and tropical geometry.

DEFINITION 2.2. The **tropicalization** function $Trop : (\mathbb{C}\{\{t\}\} \setminus \{0\})^2 \to \mathbb{R}^2$ is defined by

$$Trop(x, y) = (-val(x), -val(y)). \tag{9}$$

The connection between tropical curves and curves definied over the field of Puiseaux series is made precise by the following theorem.

THEOREM 2.1 (Kapranov). *Given* $p(x, y) = \sum c_\alpha(t) x^{\alpha_1} y^{\alpha_2} \in \mathbb{C}\{\{t\}\}[x, y]$, *define*

$$Trop(p) := \bigoplus -val(c_\alpha(t)) \odot x^{\odot \alpha_1} \odot y^{\odot \alpha_2}, \tag{10}$$

and

$$V(p) := \{(x, y) \in \mathbb{C}\{\{t\}\}^2 | p(x, y) = 0\}. \tag{11}$$

Then

$$\overline{Trop(V(p))} = V(Trop(p)), \tag{12}$$

where the overline means closure with respect to the Euclidean topology in $\mathbb{R}^2$.

The content of Kapranov's theorem is the following: you can tropicalize a polynomial, which means taking (the negative of) valuations of its coefficients and replacing ordinary operations with their tropical counterparts; and you can tropicalize sets of points in the Puiseaux plane, by taking (the negative of) valuations of their coordinates. If you start from a

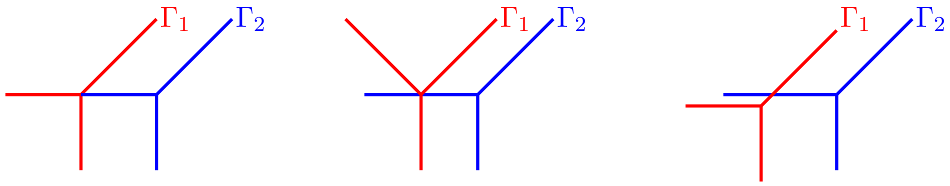


FIGURE 2.1. The leftmost tropical curves do not intersect transversely because their intersection consists of infinitely many points. The central pair also do not intersect transversely, since the intersection point is a vertex of Γ_1. The right hand side tropical curves intersect transversely.

polynomial p *with Puiseaux coefficients, and tropicalize the set of points which are solutions of* p*, you obtain a dense subset (think of "almost everything") of the tropical curve associated to the tropicalization of* p*. There are an easy and a hard direction to prove in this theorem. The next set of activities leads us to explore the easy direction.*

QUESTIONS/ACTIVITIES 2.2.

(1) Verify Theorem 2.1 in the case of a line. Consider the equation $x + y + 1 = 0$, and write a parameterization of its solution set (this just means solve for y). Then show that taking $Trop$ of the points of the solution set produces a tropical line.

(2) Given a monomial $\mu = a(t)x^m y^n$ and a point in the Puiseaux plane $Q = (x_0(t), y_0(t))$, show that
$$-val(\mu(Q)) = Trop(\mu)(Trop(Q)). \tag{13}$$

(3) Now consider a binomial $p(x,y) = \mu_1 + \mu_2$, and assume Q is a point in the Puiseaux plane such that $p(Q) = 0$. Prove that
$$-val(\mu_1(Q)) = -val(\mu_2(Q)). \tag{14}$$

(4) Based on the previous exercises, argue that for any polynomial $P \in \mathbb{C}\{\{t\}\}[x,y]$ we have
$$Trop(V(p)) \subseteq V(Trop(p)). \tag{15}$$

(5) Consider a polynomial of degree d of the form $p(x,y) = x^d + y^d + 5 + p_{d-1}(x,y)$, where p_{d-1} is a polynomial of degree $d-1$ with complex coefficients (complex numbers can also be thought as Puiseaux series!). Compute $Trop(V(p))$ without using Theorem 2.1 and verify that it does satisfy the statement of Theorem 2.1.

DEFINITION 2.3. A tropical plane curve has **degree** d if it is dual to some subdivision of the triangle Δ_d with vertices $(0,0), (d,0), (0,d)$.

We say that two tropical plane curves **intersect transversely** if they intersect in a finite number of points which are not vertices for either of the curves, as shown in Figure 2.1.

DEFINITION 2.4. Let Γ_1, Γ_2 be two tropical curves, which intersect transversely at the point P. Denote by e_1 the edge of Γ_1 containing P, $\mathbf{p_1}$ the primitive vector in the direction of e_1 and w_1 the weight of e_1, and similarly for the second curve. We define the **multiplicity of intersection** of Γ_1 and Γ_2 at P to be

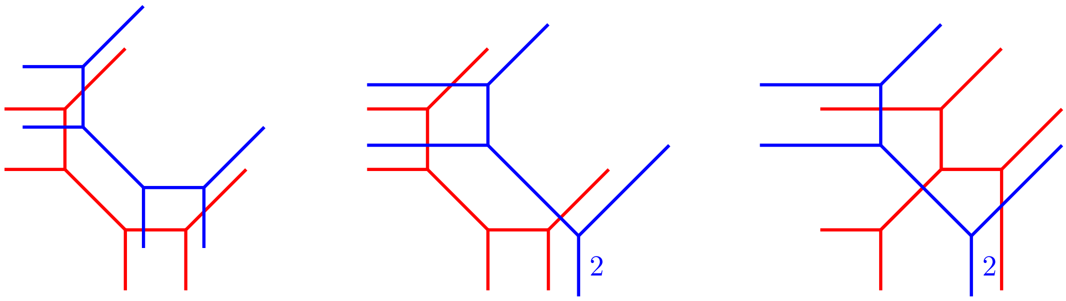
$$mult_P(\Gamma_1, \Gamma_2) := \left| det \left(\begin{bmatrix} x(w_1\mathbf{p_1}) & x(w_2\mathbf{p_2}) \\ y(w_1\mathbf{p_1}) & y(w_2\mathbf{p_2}) \end{bmatrix} \right) \right|. \tag{16}$$

THEOREM 2.2 (Bézout). *Let Γ_1 be a tropical plane curve of degree d_1 and Γ_2 a tropical plane curve of degree d_2; assume that Γ_1 and Γ_2 intersect transversely. Then:*

$$\Gamma_1 \cdot \Gamma_2 := \sum_{P \in \Gamma_1 \cap \Gamma_2} mult_P(\Gamma_1, \Gamma_2) = d_1 d_2. \tag{17}$$

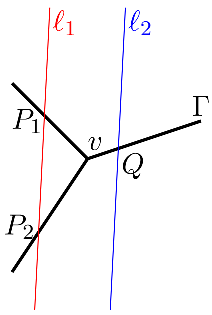
QUESTIONS/ACTIVITIES 2.3.

(1) Compute $\Gamma_1 \cdot \Gamma_2$ for the following pairs of curves, and verify the statement of Bézout's theorem.



(2) In the following picture, v is a balanced vertex of a tropical curve Γ: the three edges have weights w_i and primitive vectors $\mathbf{p_i}$ such that $\sum w_i \mathbf{p_i} = 0$. The lines ℓ_1 and ℓ_2 are parallel. Denote P_1, P_2, Q the intersection points. Prove that

$$mult_{P_1}(\Gamma, \ell_1) + mult_{P_2}(\Gamma, \ell_1) = mult_Q(\Gamma, \ell_2). \tag{18}$$



(3) Use the idea from the previous exercise to devise a strategy of proof for Bézout's theorem.

Further exercises

EXERCISE 2.1. Show that the function *val* defined in (7) is a valuation for the field of Puiseaux series.

EXERCISE 2.2. Given two integral vectors $\mathbf{v_1}, \mathbf{v_2} \in \mathbb{R}^2$, prove that

$$\left| det\left(\begin{bmatrix} x(\mathbf{v_1}) & x(\mathbf{v_2}) \\ y(\mathbf{v_1}) & y(\mathbf{v_2}) \end{bmatrix}\right)\right| = \left|\frac{\mathbb{Z}^2}{\mathbb{Z}\mathbf{v_1} + \mathbb{Z}\mathbf{v_2}}\right|. \tag{19}$$

This number is called the **lattice index** of the lattice generated by the vectors $\mathbf{v_1}, \mathbf{v_2}$ inside the lattice $\mathbb{Z}^2$.

EXERCISE 2.3. Give a complete and detailed proof of Bézout's theorem.

DAY 3

Cones and Fans

Today we start digging into the combinatorial aspects of tropical geometry. We define the notions of cone, the intersection of a finite number of half spaces, and fan, which is a collection of cones assembled in a natural way. A fan is balanced if when you "hang" it by a codimension one cone τ and pull with certain amount of force along the incident top dimensional cones, it is only allowed to slide in the τ directions. We develop a more formal definition of balancing, as well as a criterion to easily check when a fan is balanced. Finally, we develop the notion of map of fans and push-forward fan, with the desirable property that pushing forward a balanced fan yields a balanced fan.

Definition 3.1. A **rational polyhedral cone** $\sigma \subseteq \mathbb{R}^N$ can be defined in two equivalent ways:

(1) The non-negative span of a collection of vectors with rational coordinates:
$$\sigma = \left\{ \sum_{i=1}^{k} \lambda_i \mathbf{v_i} | \lambda_i \in \mathbb{R}^{\geq 0}, \mathbf{v_i} \in \mathbb{Q}^N \right\}. \tag{20}$$
(2) The intersection of a finite number of linear, rational, closed half-spaces:
$$\sigma = \bigcap_{i=1}^{l} H_i^+, \tag{21}$$
with
$$H^+ = \{\alpha_1 x_1 + \ldots + \alpha_N x_N \geq 0 | \alpha_i \in \mathbb{Q}\}. \tag{22}$$

A cone σ is called **strictly convex** there isn't any non-zero vector $\mathbf{v}$ such that $\mathbf{v}$ and $-\mathbf{v}$ both belong to σ. If we take the first perspective and think of a cone as spanned by a set of vectors, given a strictly convex cone σ there is always a unique minimal set of primitive vectors generating σ. The half-lines inside the cone spanned by these vectors are called the **rays** of σ.

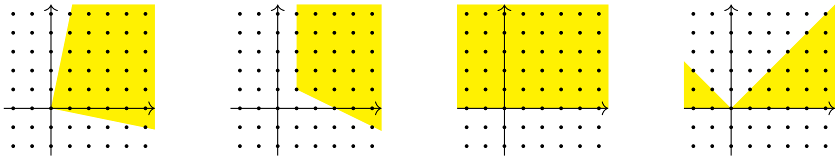
Intuitively, a strongly convex cone is a cone which is "pointy". The rays of a cone are the one-dimensional "corners" of the cone.

Definition 3.2. A **rational polyhedral fan** $\Sigma \subset \mathbb{R}^N$ is a collection of rational polyhedral cones with the property that any two cones intersect along faces:
$$\Sigma = \bigcup_{i=1}^{k} \sigma_i, \tag{23}$$
such that for any i, j, $\sigma_i \cap \sigma_j$ is a face of both σ_i and σ_j.

A **maximal cone** of Σ is any cone which is not a face of another cone of Σ. We say that a fan Σ is **pure dimensional** if all maximal cones have the same dimension. In this case, we call the dimension of maximal cones the **dimension** of Σ.

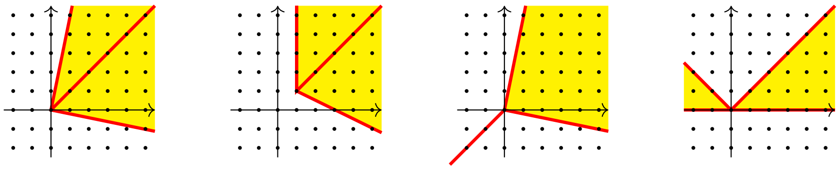
QUESTIONS/ACTIVITIES 3.1.

(1) Which of the following pictures represent a rational polyhedral cone?



(2) We did not give precise definitions of the notions of **dimension** of a cone, and **face** of a cone. Given the intuitive discussions we have had about them, try and formulate precise definitions for these concepts.

(3) Which of the following pictures represent a rational polyhedral fan? Which ones are pure dimensional?



(4) Given a rational polyhedral fan Σ, we define the **support** of Σ, denoted $|\Sigma|$, to be the set of points in $\mathbb{R}^N$ that belong to some cone of Σ. Decide which of the following statements are true:
   (a) The support of Σ is a linear subspace of $\mathbb{R}^n$.
   (b) The support of Σ is a convex subset of $\mathbb{R}^n$.
   (c) If $\mathbf{x} \in |\Sigma|$ and λ is a non-negative number, then $\lambda\mathbf{x} \in |\Sigma|$.
   (d) If $|\Sigma_1| = |\Sigma_2|$, then $\Sigma_1 = \Sigma_2$.

A **weight function** ω_Σ on a fan Σ is a function from the set of cones of Σ to the non-negative integers $\mathbb{Z}^{\geq 0}$.

Given a codimension one cone $\tau \in \Sigma$, we define a **normal vector to τ in σ**, denoted $\mathbf{u}_{\tau/\sigma}$, to be any vector in σ which descends to a generator of the lattice $\frac{Span(\sigma)\cap\mathbb{Z}^N}{Span(\tau)}$. Note that there are typically many choices for $\mathbf{u}_{\tau/\sigma} \in \mathbb{R}^N$; however, they all descend to the same vector in $\mathbb{R}^N/Span(\tau)$.

DEFINITION 3.3. A pure dimensional, rational, polyhedral fan with a weight function is **balanced** if for every codimension one face τ of Σ we have:

$$\sum_{\sigma>\tau} \omega_\Sigma(\sigma)\mathbf{u}_{\tau/\sigma} = 0 \in \mathbb{R}^N/Span(\tau). \tag{24}$$

The notation $\sigma > \tau$ means: τ is a face of σ.

DEFINITION 3.4. A **marking** on a $\mathbb{Z}$-rational, simplicial fan is a choice of an integral vector (not necessarily primitive) on each ray of the fan (see Figure 3.1). A fan with a marking is called a **marked fan**.

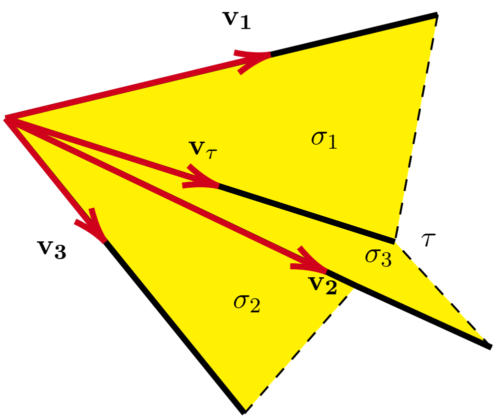


FIGURE 3.1. An illustration of a marking on a two dimensional fan.

A marking on a fan $\Sigma \subset \mathbb{R}^N$ is a way to give the fan a weight function. Given any cone $\tau \in \Sigma$, let $\rho_1, \dots, \rho_k$ be the rays bounding τ and $\mathbf{v_1}, \dots, \mathbf{v_k}$ the corresponding vectors in the marking. We define:

$$\omega_\Sigma(\tau) := \left| \frac{Span(\tau) \cap \mathbb{Z}^N}{\mathbb{Z}\mathbf{v_1} + \dots + \mathbb{Z}\mathbf{v_k}} \right|. \tag{25}$$

Equation (25) has a geometric meaning very much related to Definition 2.4: the weight of the cone σ is the index (cardinality of the quotient) of the lattice generated by the markings inside the restriction of the ambient lattice to the subvector space spanned by the cone σ. As in (16), this can be computed as the absolute value of the determinant of a matrix expressing $\mathbf{v_1}, \dots \mathbf{v_k}$ as linear combinations of a minimal set of generators of the lattice $Span(\tau) \cap \mathbb{Z}^N$.

A nice feature of weight functions induced by a marking is that one can easily check if a fan is balanced, as we show in the next Lemma.

LEMMA 3.1. *Consider a marked fan Σ, a codimension one cone $\tau \in \Sigma$ and a top dimensional cone $\sigma > \tau$. There is a unique vector in the marking that belongs to σ and does not belong to τ, denote it by $\mathbf{v}_{\sigma \smallsetminus \tau}$. Then Σ is a balanced fan if and only if*

$$\sum_{\sigma > \tau} \mathbf{v}_{\sigma \smallsetminus \tau} = 0 \in \mathbb{R}^N / Span(\tau). \tag{26}$$

PROOF. We rewrite the balancing condition (24) and show it is equivalent to (26). Let $\mathbb{R}^{K+1} \cong Span(\sigma) \supseteq \sigma > \tau \subseteq Span(\tau) \cong \mathbb{R}^K$. Up to the action of a matrix in $SL(K+1, \mathbb{Z})$, we may assume that τ is contained in the hyperplane $x_{K+1} = 0$. We then have the two following important facts:

$$\mathbf{v}_{\sigma \smallsetminus \tau} = x_{K+1}(\mathbf{v}_{\sigma \smallsetminus \tau}) \mathbf{u}_{\tau/\sigma}. \tag{27}$$

$$\omega_\Sigma(\sigma) = x_{K+1}(\mathbf{v}_{\sigma \smallsetminus \tau}) \omega_\Sigma(\tau). \tag{28}$$

We now deduce

$$\sum_{\sigma>\tau}\omega_\Sigma(\sigma)\mathbf{u}_{\tau/\sigma} = \sum_{\sigma>\tau} x^\sigma_{K+1}(\mathbf{v}_{\sigma\smallsetminus\tau})\omega_\Sigma(\tau)\frac{\mathbf{v}_{\sigma\smallsetminus\tau}}{x^\sigma_{K+1}(\mathbf{v}_{\sigma\smallsetminus\tau})} = \omega_\Sigma(\tau)\sum_{\sigma>\tau}\mathbf{v}_{\sigma\smallsetminus\tau}. \quad (29)$$

Since we assume Σ is a simplicial fan, $\omega_\Sigma(\tau)\neq 0$, which implies that

$$\sum_{\sigma>\tau}\omega_\Sigma(\sigma)\mathbf{u}_{\tau/\sigma}\in Span(\tau) \iff \sum_{\sigma>\tau}\mathbf{v}_{\sigma\smallsetminus\tau}\in Span(\tau). \quad (30)$$

□

Questions/Activities 3.2.

(1) What does the balancing condition state if Σ is a one-dimensional fan? What are the normal vectors to a codimension one face of Σ?
(2) Consider a two dimensional fan $\Sigma\subseteq\mathbb{R}^3$ and assume a portion of it looks like Figure 3.1, with $\mathbf{v}_\tau=[1,0,0], \mathbf{v_1}=[1,1,0], \mathbf{v_2}=[2,2,2]$ and $\mathbf{v_3}=[1,-3,-2]$. Compute the weights $\omega_\Sigma(\sigma_i)$ for the three top dimensional cones, compute the normal vectors $\mathbf{u}_{\tau/\sigma}$ and verify that Σ is balanced at the face τ.

Definition 3.5. Let $\Sigma_1\subseteq\mathbb{R}^M$ and $\Sigma_2\subseteq\mathbb{R}^N$. A map $f:|\Sigma_1|\to|\Sigma_2|$ is called a **map of fans** if f is the restriction of a $\mathbb{Z}$-linear map $\mathbb{R}^M\to\mathbb{R}^N$.

We would like a map of fans to be a function that sends cones to cones. Note that with this definition in place, this is not necessarily the case. However, given a map of fans, one may always subdivide some cones (both in the source and target fans) to obtain two new fans with the support, such that the same map now has the property of mapping cones to cones. So from now on when we talk about maps of fans let us assume that this additional property is verified. We define the **push-forward** of the fan Σ_1 via f to be

$$f_*(\Sigma_1) := \left\{f(\sigma)|\sigma \text{ is a maximal cone in } \Sigma_1 \text{ and } f_{|\sigma} \text{ is injective}\right\}. \quad (31)$$

If Σ_1 has a weight function, we can induce a weight function on $f_*(\Sigma_1)$ as follows. If τ is a cone of $f_*(\Sigma_1)$, we define:

$$\omega_{f_*(\Sigma_1)}(\tau) := \sum_{\tilde\tau\ s.t.\ f(\tilde\tau)=\tau}\omega_{\Sigma_1}(\tilde\tau)\left|\frac{Span(\tau)\cap\mathbb{Z}^N}{f(Span(\tilde\tau)\cap\mathbb{Z}^M)}\right|. \quad (32)$$

Example 3.1. Consider the fans $\Sigma_1\subseteq\mathbb{R}^2$, $\Sigma_2\subseteq\mathbb{R}$ as depticted in Figure 3.2; all cones have weight one. In the first case, $f(x,y)=y$ is horizontal projection. Using (32), we have:

$$\omega_{f_*(\Sigma_1)}(\sigma_1) = \left|\frac{Span(\sigma_1)\cap\mathbb{Z}}{f(Span(\tilde\sigma_1)\cap\mathbb{Z}^2)}\right| = \left|\frac{\mathbb{Z}\cdot 1}{\mathbb{Z}\cdot f([1,1])}\right| = 1, \quad (33)$$

$$\omega_{f_*(\Sigma_1)}(\sigma_2) = \left|\frac{Span(\sigma_2)\cap\mathbb{Z}}{f(Span(\tilde\sigma_2)\cap\mathbb{Z}^2)}\right| = \left|\frac{\mathbb{Z}\cdot(-1)}{\mathbb{Z}\cdot f([0,-1])}\right| = 1. \quad (34)$$

We conclude that $f_*(\Sigma_1)=\Sigma_2$.

Now consider the function $g(x,y)=x+y$. Now we have

$$\omega_{g_*(\Sigma_1)}(\sigma_1) = \left|\frac{Span(\sigma_1)\cap\mathbb{Z}}{g(Span(\tilde\sigma_1)\cap\mathbb{Z}^2)}\right| = \left|\frac{\mathbb{Z}\cdot 1}{\mathbb{Z}\cdot g([1,1])}\right| = 2, \quad (35)$$

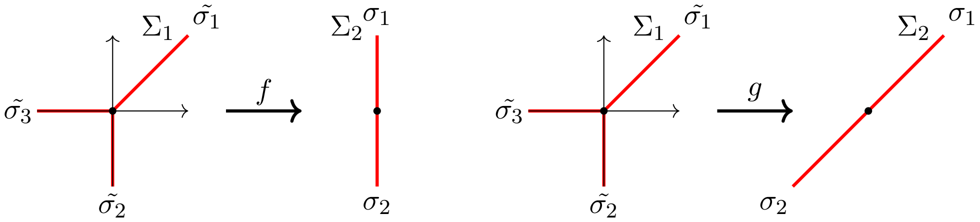


FIGURE 3.2. Two examples of maps of fans.

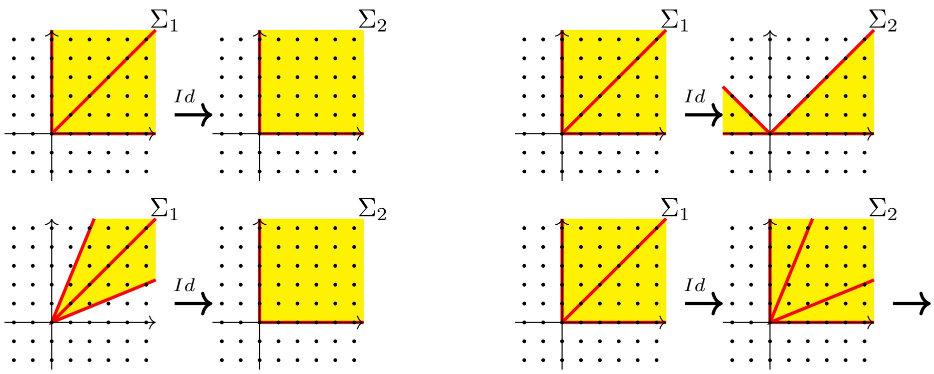
$$\begin{aligned}\omega_{g_*(\Sigma_1)}(\sigma_2) &= \left|\frac{Span(\sigma_2)\cap\mathbb{Z}}{g(Span(\tilde{\sigma}_2)\cap\mathbb{Z}^2)}\right| + \left|\frac{Span(\sigma_2)\cap\mathbb{Z}}{g(Span(\tilde{\sigma}_3)\cap\mathbb{Z}^2)}\right| \\ &= \left|\frac{\mathbb{Z}\cdot(-1)}{\mathbb{Z}\cdot g([0,-1])}\right| + \left|\frac{\mathbb{Z}\cdot(-1)}{\mathbb{Z}\cdot g([-1,0])}\right| = 1+1. \qquad (36)\end{aligned}$$

We conclude that in this case $g_*(\Sigma_1) = 2\Sigma_2$.

LEMMA 3.2. *Let $\Sigma_1 \subseteq \mathbb{R}^M$ be a balanced fan, and f a map of fans. Then $f_*(\Sigma_1)$ is a balanced fan.*

QUESTIONS/ACTIVITIES 3.3.

(1) For each of the pictures below, consider the identity function $Id : \mathbb{R}^2 \to \mathbb{R}^2$. Decide if Id induces a map of fans as in Definition 3.5. If it does, show how the cones of Σ_1 and Σ_2 should be subdivided in order for the map of fans to send cones to cones.



(2) If Σ_1 is a pure dimensional fan of dimension k, what is the dimension of $f_*(\Sigma)$?
(3) Consider the fan $\Sigma_1 \subseteq \mathbb{R}^2$, consisting of four rays generated by $\pm\mathbf{e_1}, \pm\mathbf{e_2}$. What are the conditions on the weights on the four rays for Σ_1 to be a balanced fan? Now consider the map $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x,y) = 2x + 3y$. Describe the fan $f_*(\Sigma_1)$ and check it is balanced.

DEFINITION 3.6. A balanced fan $\Sigma \subseteq \mathbb{R}^M$ is **irreducible** if it cannot be decomposed as the sum of two balanced fans with different supports. Here

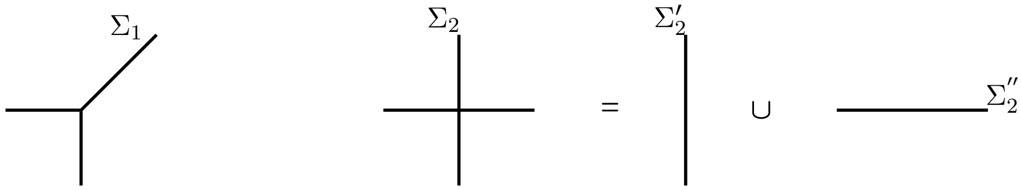


FIGURE 3.3. The fan Σ_1 is irreducible. The fan Σ_2 is reducible.

by sum of two balanced fans we mean taking the union of the cones in each fan, and adding the weights for any cone that appears in both fans.

EXAMPLE 3.2. Consider the two fans in Figure 3.3, where all cones have weight one. The fan Σ_1 is irreducible, while Σ_2 is not, as it may be decomposed as the sum of the two subfans $\Sigma_2' \cup \Sigma_2''$.

LEMMA 3.3. *Suppose Σ_1, Σ_2 are two balanced fans of the same dimension, Σ_1 is irreducible, and $|\Sigma_2| \subseteq |\Sigma_1|$. Then, up to subdivision, there exists a positive rational number λ such that*

$$\Sigma_1 = \lambda \Sigma_2. \tag{37}$$

We now come to the definition of the degree of a map of balanced fans.

DEFINITION 3.7. Let $\Sigma_1 \subseteq \mathbb{R}^M, \Sigma_2 \subseteq \mathbb{R}^N$ be two balanced fans of the same dimension, $f : \Sigma_1 \to \Sigma_2$ a map of fans.

$$mult_P(f) := \frac{\omega_{\Sigma_1}(\sigma_P)}{\omega_{\Sigma_2}(\sigma_{f(P)})} \left| \frac{Span(\sigma_{f(P)}) \cap \mathbb{Z}^N}{f(Span(\sigma_P) \cap \mathbb{Z}^M)} \right|. \tag{38}$$

Next assume Σ_2 is irreducible. We define the **degree** of f. For any point Q in the interior of a maximal cone of Σ_2:

$$deg(f) := \sum_{P \ s.t. \ f(P)=Q} mult_P(f). \tag{39}$$

The next theorem shows that the degree of f is well-defined.

THEOREM 3.1. *If Σ_2 is an irreducible fan, and $Q, \tilde{Q}$ are two points in the interior of maximal cones of Σ_2, then*

$$\sum_{P \ s.t. \ f(P)=Q} mult_P(f) = \sum_{P \ s.t. \ f(P)=\tilde{Q}} mult_P(f). \tag{40}$$

PROOF. Cosider the weighted fan $f_*(\Sigma_1)$. Since its support is contained in Σ_2 which is irreducible, by Lemma 3.3 there exists a rational number λ such that

$$\lambda \Sigma_2 = f_*(\Sigma_1). \tag{41}$$

This implies that for any point Q in the interior of a maximal cone of Σ_2,

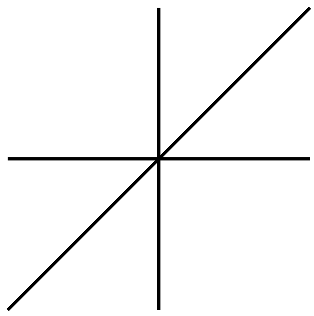
$$\lambda \omega_{\Sigma_2}(\sigma_Q) = \omega_{f_*(\Sigma_1)}(\sigma_Q) = \sum_{P \ s.t. \ f(P)=Q} \omega_{\Sigma_1}(\sigma_P) \left| \frac{Span(\sigma_Q) \cap \mathbb{Z}^N}{f(Span(\sigma_P) \cap \mathbb{Z}^M)} \right|, \tag{42}$$

where the last equality is (32). The theorem is proved by dividing by $\omega_{\Sigma_2}(\sigma_Q)$ and observing that λ, which is independent of Q, equals the definition of $deg(f)$.

□

QUESTIONS/ACTIVITIES 3.4.

(1) Show that the fan in the picture below is not irreducible, and further that it may be decomposed as a sum of balanced fans in more than one way.



(2) Prove Lemma 3.3.

REMARK 3.1 (Very Important!). Assume Σ_1, Σ_2 are marked fans, denote $v_1, \ldots, v_n$ the marking on the rays of a top dimensional cone $\sigma_P \in \Sigma_1$ and by $w_1, \ldots, w_n$ the marking on the rays of $\sigma_Q = f(\sigma_P) \in \Sigma_2$. Then

$$mult_P(f) = det(M_f), \tag{43}$$

where M_f is the matrix representing the linear function f in the bases $v_1, \ldots, v_n$ and $w_1, \ldots, w_n$.

To see that this is true, pick orthonormal bases β_1 for $Span(\sigma_P)$ and β_2 for $Span(\sigma_Q)$, and denote by $M^{\mathbf{v}}_{\beta_1}$ and $M^{\beta_2}_{\mathbf{w}}$ the matrices of the changes of bases with respect to the bases given by the markings. Also denote by $\widetilde{M}_f$ the matrix representing f in the two bases β_1 and β_2. Then we have:

$$M_f = M^{\beta_2}_{\mathbf{w}} \widetilde{M}_f M^{\mathbf{v}}_{\beta_1}, \tag{44}$$

which implies

$$det(M_f) = det(M^{\beta_2}_{\mathbf{w}}) det(\widetilde{M}_f) det(M^{\mathbf{v}}_{\beta_1}). \tag{45}$$

The remark now follows by observing that

$$det(M^{\beta_2}_{\mathbf{w}}) = \frac{1}{\omega_{\Sigma_2}(\sigma_Q)}, \quad det(\widetilde{M}_f) = \left| \frac{Span(\sigma_Q) \cap \mathbb{Z}^N}{f(Span(\sigma_P) \cap \mathbb{Z}^M)} \right|, \quad det(M^{\mathbf{v}}_{\beta_1}) = \omega_{\Sigma_1}(\sigma_P). \tag{46}$$

Further exercises

EXERCISE 3.1. Prove Lemma 3.2. Focus on a codimension one face τ and a top dimensional face $\sigma > \tau$ in $f_*(\Sigma)$, and on a face $\tilde{\sigma} \in \Sigma$ such that $f(\tilde{\sigma}) = \sigma$. You may assume that you use a transformation in $SL(Span(\sigma), \mathbb{Z})$ so that τ lives in the hyperplane where the last coordinate is zero. Then compute the contribution by $\tilde{\sigma}$ to the weight of σ in $f_*(\Sigma)$, and compute the relationship between the vector $\mathbf{u}_{\tau/\sigma}$ and $f(\mathbf{u}_{\tilde{\tau}/\tilde{\sigma}})$.

EXERCISE 3.2. Let $\Sigma \subseteq \mathbb{R}^3$ be a pure two dimensional fan whose maximal cones consist of the twelve coordinate orthants of $\mathbb{R}^3$. Choose a marking on this fan that makes it into a balanced fan. What are the weights of the maximal cones with your choice of marking? Consider the linear function $f : \mathbb{R}^3 \to \mathbb{R}^2$ given by $f(x, y, z) = 2y + 3z$. Describe the fan $f_*(\Sigma)$.

EXERCISE 3.3. Suppose $p \in \mathbb{C}\{\{t\}\}[x, y]$ and $\overline{Trop(V(p))}$ equals the reducible fan Σ_2 in Figure 3.3. Does this imply that the polynomial p must be a reducible polynomial?

DAY 4

Abstract tropical curves and their moduli

Today we introduce the notion of an abstract tropical curve: this is a metric tree, where we do not worry about it living inside any particular ambient vector space. We construct a parameter space for the set of all trees with a fixed number of labeled ends; it naturally consists of a collection of cones glued along their faces. The punchline of today is that we can realize this space inside a very large vector space as a balanced fan, where all cones have weight 1*. This is called the moduli space of abstract, tropical, n-pointed, rational curves and is denoted $M_{0,n}^{trop}$.*

DEFINITION 4.1. An **abstract, rational, n-pointed tropical curve** is a tree T with n labeled ends, and a function (called the **metric**) from the set of edges $m : E(T) \to \mathbb{R}^{\geq 0}$. A curve is called **stable** if each vertex has valence ≥ 3.

We consider ends to be unbounded edges. In particular, there is only one vertex adjacent to an end. See Figure 4.1 for some examples.

If you start from the tropicalization of a rational curve, which gives you a tropical curve in the plane, you get an abstract tropical curve by forgetting the embedding but remembering the length of all finite edges of the tropical curve. We allow the metric function to take value 0*, with the assumption that we declare a graph with an edge of length zero equivalent to the graph obtained by contracting that edge.*

Forgetting the information of the metric for a tropical curve Γ, we obtain a stable tree, called the **topological type** of Γ.

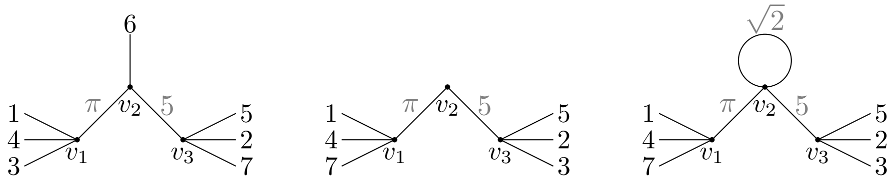


FIGURE 4.1. The first picture is an example of a stable, abstract, rational, tropical curve, with three vertices, two edges and seven labeled ends. The length of the edges is written in gray. The second picture is not stable because v_1 has valence 2. The third picture is not rational, as it has a loop starting and ending at v_2.

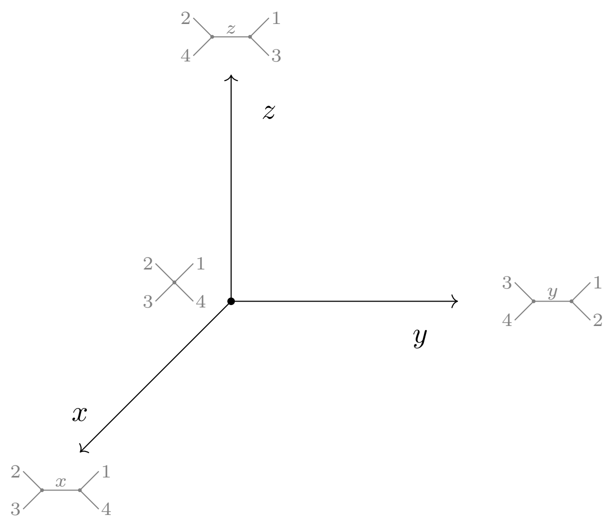


FIGURE 4.2. The cone complex $M_{0,4}^{trop}$ is obtained by identifying the vertices of three one dimensional cones (rays). The picture illustrates the cone complex, and in gray the tropical curves parameterized by each ray.

QUESTIONS/ACTIVITIES 4.1.

(1) What are the topological types of abstract, rational, stable, n-pointed tropical curves, for $n = 3, 4, 5$?
(2) What are the minimum and maximum number of compact edges that an abstract, rational, stable, n-pointed tropical curve can have?

Let T be a stable tree with n-ends and m edges. The set of all tropical curves of topological type T, or equivalently, the set of all possible metrizations of the edges of T, is naturally parameterized by the cone $(\mathbb{R}^{\geq 0})^m =: C_T$.

DEFINITION 4.2. We denote by $M_{0,n}^{trop}$ the parameter space for stable, abstract, rational, n-pointed tropical curves. It consists of the cone complex

$$\coprod_T C_T \Big/ \sim, \tag{47}$$

where the disjoint union is over all topological types of stable trees with n ends, and two points $[\Gamma] \in C_T$, $[\Gamma'] \in C_{T'}$ are identified if $\Gamma \sim \Gamma'$, i.e. if they are equal after contracting all edges with length 0.

The space $M_{0,4}^{trop}$ is illustrated in Figure 4.2.
The **tropical forgetful morphim**

$$\pi_{n+1}^{trop} : M_{0,n+1}^{trop} \to M_{0,n}^{trop}$$

assigns to a graph with $(n+1)$-ends Γ a graph Γ' obtained by forgetting the end labeled $(n+1)$ and *stabilizing* the result if needed. This means that if a 2-valent vertex v is formed when forgetting $(n+1)$, it should be demoted to an ordinary point. Then:

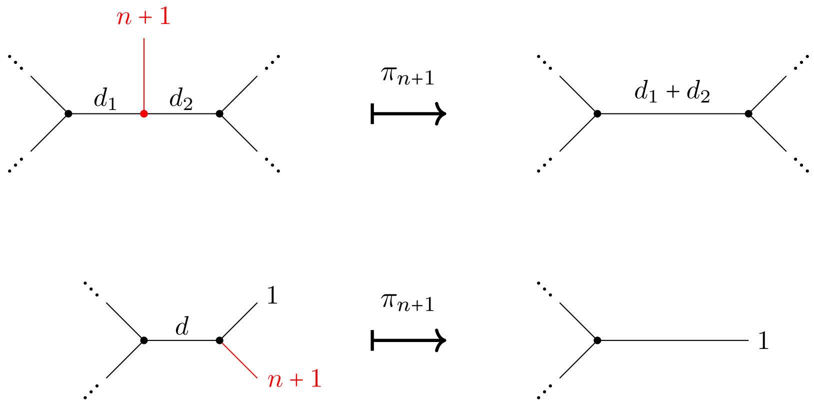


FIGURE 4.3. The process of stabilizing a graph after forgetting the $(n+1)$-th end.

(1) If v separated two edges of length d_1 and d_2, we now have only one edge of length $d_1 + d_2$;
(2) If v was adjacent to an edge and an end, then only the end is left.

See Figure 4.3 for an illustration.

QUESTIONS/ACTIVITIES 4.2.

(1) Describe the space $M_{0,3}^{trop}$.
(2) Understand that the space $M_{0,5}^{trop}$ is two dimensional, and it is (combinatorially) represented as the cone over the Petersen graph, as shown in Figure 4.4. Label the two-dimensional cones in Figure 4.4 by the topological types of the curves that they parameterize.
(3) What is the dimension of $M_{0,n}^{trop}$? Give a combinatorial description of the tropical curves parameterized by the top dimensional cones, and by the codimension one cones.
(4) Understand the forgetful morphism $\pi_5 : M_{0,5}^{trop} \to M_{0,4}^{trop}$.
(5) Show that the tropical forgetful morphisms map cones to cones. Characterize which cones it is bijective on, and which cones it contracts to lower dimensional cones.

Now we define an embedding of $M_{0,n}^{trop}$ into $\mathbb{R}^{\binom{n}{2}}$ as follows:

$$\begin{array}{rccc} dist: & M_{0,n}^{trop} & \to & \mathbb{R}^{\binom{n}{2}} \\ & [\Gamma] & \mapsto & (d(i,j))_{(i,j)} \end{array},$$

where $d(i,j)$ denotes the distance between the i-th and the j-th end of the tropical curve using the edge metric.We will check that this is a piecewise linear embedding of $M_{0,n}^{trop}$, and see the image cannot be made into a balanced fan (this already fails at $n = 4$). So we take a quotient of $\mathbb{R}^{\binom{n}{2}}$.

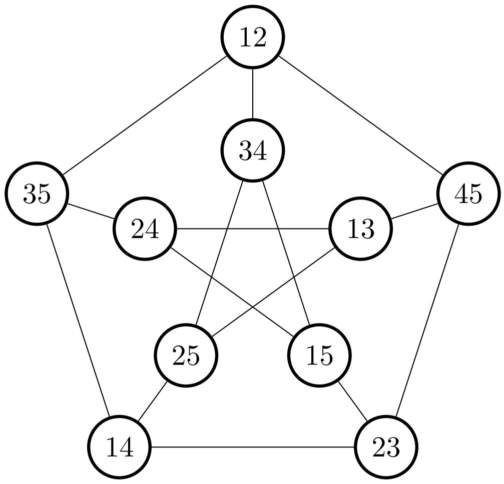


FIGURE 4.4. The cone complex $M_{0,5}^{trop}$ is the cone over the Petersen graph depicted here. Each ray parameterizes a tropical curve with one edge, two ends on one side, and three on the other. On the corresponding vertex of the Petersen graph we have marked the labels of the two ends that are together in the tropical curve.

Define the linear function:

$$\begin{array}{cccc} \Phi: & \mathbb{R}^n & \to & \mathbb{R}^{\binom{n}{2}} \\ & (a_1,\dots,a_n) & \mapsto & (a_i+a_j)_{(i,j)} \end{array},$$

and define the quotient space $Q = \mathbb{R}^{\binom{n}{2}}/Im(\Phi)$.

For any subset $I \subset [n]$ such that $2 \leq |I| \leq n-2$, consider the tropical curve Γ_I consisting of two vertices joined by one edge of length one, the marks in I attached to one vertex, and the marks in I^c attached to the other. We define

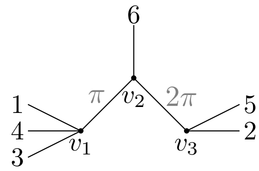
$$\mathbf{v}_I := dist(\Gamma_I) \subset \mathbb{R}^{\binom{n}{2}}. \tag{48}$$

*We sometimes will want to think of $\mathbf{v}_I$ as living in the quotient space Q; we will let the context make that distinction. To avoid too much notation, we will also call **dist** the composition of the map **dist** with the projection function to Q. The map $dist : M_{0,n}^{trop} \to Q$ is still an injective function. Here we are going to take it for granted.*

QUESTIONS/ACTIVITIES 4.3.

(1) Study the function $dist : M_{0,4}^{trop} \to \mathbb{R}^6$. Show that it is injective, and linear on each cone of $M_{0,4}^{trop}$. Show the image of $dist$ cannot be made into a balanced fan in $\mathbb{R}^6$. Show that $\mathbf{v}_{\{12\}} + \mathbf{v}_{\{13\}} + \mathbf{v}_{\{14\}}$ lies in the image of Φ and conclude that $M_{0,4}^{trop}$ can be made into a balanced fan in Q.

(2) Consider the tropical curve $[\Gamma] \in M_{0,6}^{trop}$ depicted in the following picture. Write down the coordinates of the vectors $dist([\Gamma]), \mathbf{v}_{134}, \mathbf{v}_{25}$ in the table. Are the three vectors linearly independent?



	(12)	(13)	(14)	(15)	(16)	(23)	(24)	(25)	(26)	(34)	(35)	(36)	(45)	(46)	(56)
$dist([\Gamma])$															
$\mathbf{v}_{\{134\}}$															
$\mathbf{v}_{\{25\}}$															

(3) Let Γ be an abstract, rational, tropical, n-pointed curve. Each edge e of Γ defines a two-part partition $[n] = I_e \cup I_e^c$ of the set of indices, by considering the indices that lie on either side of the edge. If we denote by $l(e)$ the length of the edge e, show that we have:

$$dist(\Gamma) = \sum_{e \in \Gamma} l_e \mathbf{v}_{I_e}. \tag{49}$$

(4) Given a topological type T, consider the cone $C_T \cong (\mathbb{R}^{\geq 0})^m$ and denote by $\mathbf{e_i}$ the standard basis vector corresponding to the i-th edge e_i. Show that

$$dist(\mathbf{e_i}) = \mathbf{v}_{I_{e_i}}. \tag{50}$$

(5) If Γ belongs to the interior of a cone $\tau = C_T$ of $M_{0,n}^{trop}$, show that if the $\mathbf{v}_{I_e}$ from the previous exercises are choosen as the markings on the rays of τ, then the weight of τ is equal to 1.

Theorem 4.1. *The image of $dist(M_{0,n}^{trop})$, with all cones taken with weight one, is a balanced, marked fan in Q.*

Let us prove this theorem together, in the next group of activities.

Questions/Activities 4.4.

(1) Recall Lemma 3.1, and spell it out in this particular case. In particular, what does it mean for the sum of the $\mathbf{v}_{\sigma \setminus \tau}$ to live in the span of τ inside Q?
(2) Consider a codimension one cone τ: it must parameterize curves with exactly one four-valent vertex, and all other vertices trivalent. Call A, B, C, D the four components of the graph attached to the four-valent vertex. Note that A, B, C, D are either ends or trivalent trees. Describe the top dimensional cones σ that contain τ as a face.
(3) Let $\tilde{\Gamma}$ be a curve obtained by starting by any curve in τ, contracting all edges not adjacent to the four-valent vertex, and giving length

one to the surviving edges. Note that $\tilde{\Gamma}$ may have at most four edges, but it may have fewer. Why?

(4) Let $\mathbf{x} = (x_1, \dots, x_n)$ be defined by $x_i = 1$ if the mark i is adjacent to the four valent vertex for a general curve in τ, and $x_i = 0$ otherwise. Describe $\Phi(\mathbf{x})$.

(5) Show that

$$\mathbf{v}_{A\cup B} + \mathbf{v}_{A\cup C} + \mathbf{v}_{A\cup D} = dist(\tilde{\Gamma}) + \Phi(\mathbf{x}) \in \mathbb{R}^{\binom{n}{2}}. \tag{51}$$

You may check that (51) holds by checking it for every coordinate of $\mathbb{R}^{\binom{n}{2}}$.

(6) Argue that you just proved Theorem 4.1! Celebrate!

From now on by $M_{0,n}^{trop}$ we will mean the balanced fan in Q thus obtained.

Further exercises

EXERCISE 4.1. Show that for $n \geq 4$:

$$\sum_{|I|=2, 1\notin I} \mathbf{v}_I = \Phi(1, n-3, n-3, \dots, n-3). \tag{52}$$

EXERCISE 4.2. Prove that the map $dist : M_{0,n}^{trop} \to Q$ is an injective function.

EXERCISE 4.3. Verify that the morphism π_{n+1}^{trop} functions as a universal family in the sense that:

$$\pi_{n+1}^{trop^{-1}}([\Gamma]) \cong \Gamma \subseteq M_{0,n+1}^{trop}.$$

Further, prove that π_{n+1}^{trop} is a map of fans, i.e. it is induced by a linear function.

EXERCISE 4.4. Show that $M_{0,n}^{trop}$ is connected through codimension one, i.e. that given any two points P, Q living in two top dimensional cones, there exists a path γ from P to Q that lives entirely in the interior of top dimensional and codimension one cones. Equivalently, this means that $M_{0,n}^{trop}$ remains connected after you remove all cones of codimension greater than one.

DAY 5

Moduli Spaces of Tropical Stable Maps

Today we introduce the moduli spaces of tropical stable maps: abstract tropical curves toghether with a function that maps them into the plane in a nice way. We will learn that these moduli spaces are quite similar to our old friends $M^{trop}_{0,n}$'s. In particular, they also are balanced fans. These moduli spaces admit evaluation functions, that allow us to identify where in the plane certain points on our tropical curves are mapped. By combining several evaluation functions, we can construct a map of balanced fans whose degree computes the number of tropical curves of a certain degree that pass through a certain number of fixed points in the plane.

Definition 5.1. A **tropical, rational, n-marked stable map to the plane** consists of a tuple (Γ, φ), where:

- $\Gamma \in M^{trop}_{0,n+m}$ is an abstract, stable, tropical, rational curve with $n+m$ ends, with $n \geq 0$ and $m \geq 2$;
- $\varphi : \Gamma \to \mathbb{R}^2$ is a continuous function which restricts to an integral affine linear function on each edge or end of Γ (i.e. if t is the length coordinate on the edge, the restriction of φ is of the form $\varphi(t) = \mathbf{v}t + \mathbf{a}$, with $\mathbf{v} \in \mathbb{Z}^2$).

We call $\mathbf{v}$ as above the **direction vector** of the edge or end. Note that in fact the sign of the direction vector depends on the choice of an orientation for the edge. We also have the following requirements:

balancing: for every vertex v of Γ, the sum of the direction vectors of the adjacent edges/ends, oriented outgoing from the vertex, is $\mathbf{0}$:

$$\sum_{e \ni v} \mathbf{v}_e = \mathbf{0}. \tag{53}$$

marks: the direction vector for each of the first n-ends equals $\mathbf{0}$.

ends: the direction vectors for the last m-ends are different from $\mathbf{0}$.

We see an example of a tropical stable maps in Figure 5.1. For simplicity of notation, we often draw a tropical stable maps just by drawing the image of the map, and we omit the labeling of the m ends that are not contracted.

Remark 5.1. Let (Γ, φ) be a tropical stable map. We may give weights to the edges of $\varphi(\Gamma)$ as follows: if $\mathbf{v_e} = (x_e, y_e)$ is the direction vector associated to an edge e, we define the weight $w_e = gcd(x_e, y_e)$. With this convention, the image curve $\varphi(\Gamma)$ satisfies the balancing condition as stated in Definition 1.7.

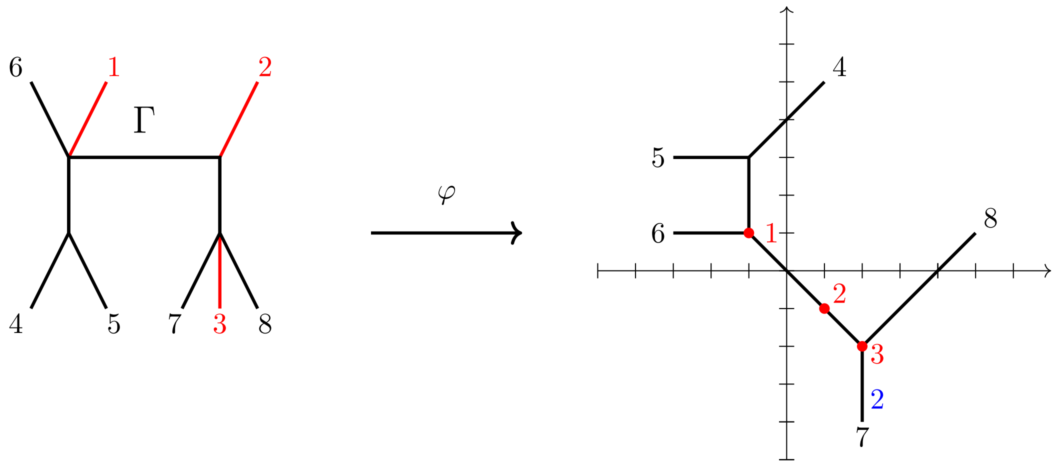


FIGURE 5.1. An example of a tropical stable map. The red marked ends are contracted to points. The blue 2, which can be thought as a weight on an end, reminds us that the direction vector for that end is twice the primitive vector in that direction.

DEFINITION 5.2. The ordered list of non-zero direction vectors of the ends of Γ is called the **degree** of the map φ and denoted by Δ.

DEFINITION 5.3. We denote by $M^{trop}_{0,n}(\mathbb{R}^2, \Delta)$ the parameter space of tropical rational, n-marked stable maps to the plane of degree Δ. In particular, denoting by $\mathbf{e_1}$ and $\mathbf{e_2}$ the standard basis of $\mathbb{R}^2$, if Δ consists of d copies of the vector $-\mathbf{e_1}$, d copies of the vector $-\mathbf{e_2}$ and d copies of the vector $\mathbf{e_1} + \mathbf{e_2}$, then we say the maps have degree d, and the target is tropical $\mathbb{P}^2$, and denote the space $M^{trop}_{0,n}(\mathbb{P}^2, d)$.

QUESTIONS/ACTIVITIES 5.1.

(1) What is the degree of the tropical stable map in Figure 5.1? What moduli space does it belong to?
(2) Describe explicitly $M^{trop}_{0,0}(\mathbb{P}^2, 1)$ and $M^{trop}_{0,1}(\mathbb{P}^2, 1)$.

DEFINITION 5.4. For $i = 1, \ldots, n$ we define the i-th **evaluation morphism**

$$ev_i : M^{trop}_{0,n}(\mathbb{R}^2, \Delta) \to \mathbb{R}^2 \tag{54}$$

by

$$ev_i(\Gamma, \varphi) := \varphi(i), \tag{55}$$

where i denotes the i-th marked end (which is contracted to a point by φ).

THEOREM 5.1. *Denote by $m = |\Delta|$. Then the function*

$$s \times ev_1 : M^{trop}_{0,n}(\mathbb{R}^2, \Delta) \to M^{trop}_{0,n+m} \times \mathbb{R}^2 \tag{56}$$

defined by

$$(\Gamma, \varphi) \mapsto (\Gamma, \varphi(1)) \tag{57}$$

is a bijection.

This theorem allows us to identify the moduli space of tropical stable maps with the product of a moduli space of tropical curves times a vector space. In particular, since we showed that $M_{0,n}^{trop}$ is a balanced fan, so is the moduli space of stable maps!

Proof. The theorem is proven by explicitly constructing an inverse function. Given a curve $\Gamma \in M_{0,n+m}^{trop}$, and a point $P \in \mathbb{R}^2$, we wish to construct a tropical stable map $\varphi : \Gamma \to \mathbb{R}^2$ in $M_{0,n}^{trop}(\mathbb{R}^2, \Delta)$. Note that the direction vectors for the last m ends are specified by Δ. The balancing condition at all vertices uniquely determines all other direction vectors. Therefore φ is determined up to a global translation in $\mathbb{R}^2$. Imposing that the vertex adjacent to the first mark maps to P fixes uniquely such translation, and yields a unique map φ. □

Evaluation functions are linear functions when restricted to cones τ of $M_{0,n}^{trop}(\mathbb{R}^2\Delta) \cong M_{0,n+m}^{trop} \times \mathbb{R}^2$. For this purpose, recall from (49) that we can use the vectors $\mathbf{v}_{I_e}$ as a basis for $Span(\tau) \subset Q$ and the length of edges l_e as dual coordinates. Then, for any $(\Gamma,(x,y)) \in \tau \times \mathbb{R}^2$, let $P_{1\to i}$ be the unique oriented path from the vertex adjacent to the first mark to the vertex adjacent to the i-th mark; then we have

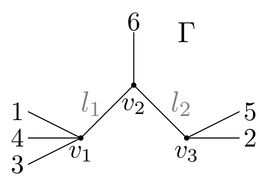
$$ev_i(\Gamma,(x,y)) = \begin{bmatrix} x \\ y \end{bmatrix} + \sum_{e \in P_{1\to i}} l_e \mathbf{v}_{\mathbf{e}}. \tag{58}$$

In fact a stronger statement holds. We state it here and leave the proof as an exercise.

Lemma 5.1. *The maps $ev_i : M_{0,n}^{trop}(\mathbb{R}^2, \Delta) \to \mathbb{R}^2$ are restriction of integral linear functions $L_i : Q \times \mathbb{R}^2 \to \mathbb{R}^2$.*

Questions/Activities 5.2.

(1) Using the tacks on the axes as units, compute $ev_1(\Gamma,\varphi), ev_2(\Gamma,\varphi), ev_3(\Gamma,\varphi)$ for the stable map in Figure 5.1.
(2) Consider the abstract tropical curve in Figure 5.1, and the point $(0,0) \in \mathbb{R}^2$. Construct the tropical stable map of degree Δ, where $\mathbf{v}_4 = (-1,0), \mathbf{v}_5 = (-1,0)\mathbf{v}_6 = (1,1), \mathbf{v}_7 = (1,1), \mathbf{v}_8 = (0,-2)$, associated to $(\Gamma,(0,0))$ via the inverse of the function $s \times ev_1$ defined in Theorem 5.1.
(3) Let $\Delta = \{(-1,0),(0,-1),(1,0),(0,1)\}$. Consider the cone $\tau \times \mathbb{R}^2$ of $M_{0,2}^{trop}(\mathbb{R}^2, \Delta) \cong M_{0,6}^{trop} \times \mathbb{R}^2$ identified by the curve Γ pictured below. Recall that $\mathbf{v}_{\{134\}}, \mathbf{v}_{\{25\}}$ give a basis for the cone τ. Write down the matrices for the functions ev_1, ev_2 using this basis and the standard basis for (both copies of) $\mathbb{R}^2$.



Now we consider the special case in which $n = |\Delta| - 1$. Observe that in this case the dimension of $M^{trop}_{0,n}(\mathbb{R}^2, \Delta)$ is equal to $2(|\Delta| - 1)$. We now consider the linear function

$$Ev := ev_1 \times \ldots \times ev_n : M^{trop}_{0,n}(\mathbb{R}^2, \Delta) \to \mathbb{R}^2 \times \ldots \times \mathbb{R}^2. \tag{59}$$

Let us believe for now that Ev preserves the dimension of all top dimensional cones of $M^{trop}_{0,n}(\mathbb{R}^2, \Delta)$. Therefore $Ev : M^{trop}_{0,n}(\mathbb{R}^2, \Delta) \to Ev_*(M^{trop}_{0,n}(\mathbb{R}^2, \Delta))$ is a map of balanced fans, which in particular implies that Ev is surjective. By Definition 3.7 we know that we have a well defined notion of degree of Ev, and Remark 3.1 tells us how to compute it!

Definition 5.5. Given any degree Δ, and all notation as above,

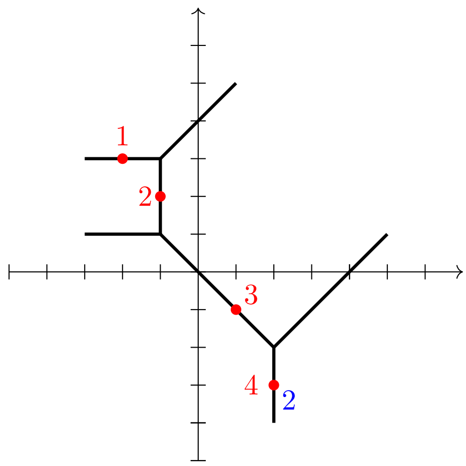
$$N^{trop}_{\Delta} := deg(Ev), \tag{60}$$

gives the (weighted) number of tropical maps of degree Δ whose image in $\mathbb{R}^2$ is a tropical curve passing through $|\Delta| - 1$ points in general position in the plane.

Questions/Activities 5.3.

(1) Spend a little time becoming comfortable with Definition 5.5. Why did we make this definition?
(2) Consider the tropical stable map $(\Gamma, \varphi) \in M^{trop}_{0,5}(\mathbb{R}^2, \Delta)$ depicted below. Compute

$$mult_{(\Gamma,\varphi)}(Ev). \tag{61}$$



(3) Compute $deg(Ev)$ for the moduli space $M^{trop}_{0,2}(\mathbb{P}^2, 1)$. What is the geometric statement you just made?

Further exercises

Exercise 5.1. Prove Lemma 5.1. In particular, using the natural coordinates z_{ij} for $\mathbb{R}^{\binom{n}{2}}$, show that

$$ev_1(\{z_{ij}\}, x, y) = \begin{bmatrix} x \\ y \end{bmatrix}, \tag{62}$$

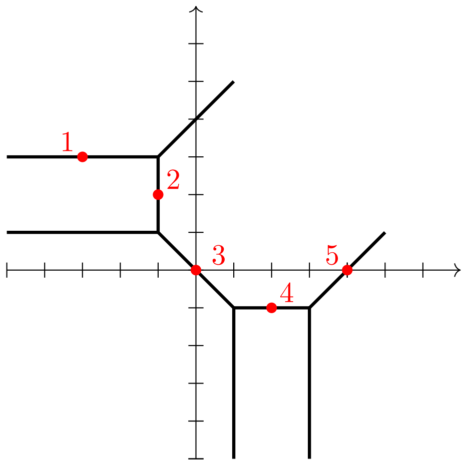
and, for $i \neq 1$ Prove Lemma 5.1. In particular, using the natural coordinates z_{ij} for $\mathbb{R}^{\binom{n}{2}}$, show that

$$ev_1(\{z_{ij}\}, x, y) = \begin{bmatrix} x \\ y \end{bmatrix} + \frac{1}{2} \sum_{\substack{k=2 \\ k \neq i}}^{n+|\Delta|} (z_{1k} - z_{ik}) \mathbf{v_k}, \tag{63}$$

where $\mathbf{v_k}$ is the direction vector assigned to the end marked by k by the direction vector Δ.

EXERCISE 5.2. Consider the tropical stable map $(\Gamma, \varphi) \in M_{0,5}^{trop}(\mathbb{P}^2, 2)$ depicted below. Compute

$$mult_{(\Gamma,\varphi)}(Ev). \tag{64}$$



EXERCISE 5.3. Repeat the computation from Exercise 5.2, but pick a different bijection $M_{0,5}^{trop}(\mathbb{P}^2, 2) \to M_{0,11}^{trop} \times \mathbb{R}^2$. Instead of using ev_1 in the bijection from Theorem 5.1, use ev_3. Observe that the matrix you use to compute the multiplicity $mult_{(\Gamma,\varphi)}(Ev)$ now is obtained from the one you computed in Exercise 5.2 by column operations, and therefore the two matrices have the same determinant. Convince yourself that this is always the case, i.e. the multiplicity of Ev at a point is independent of which evaluation function we use to fix the bijection $s \times ev_i : M_{0,n}^{trop}(\mathbb{R}^2, \Delta) \to M_{0,n+|\Delta|}^{trop} \times \mathbb{R}^2$.

DAY 6

Kontsevich/Mikhalkin's theorem

Today we are finally proving the theorem we have been aiming for!

THEOREM 6.1 (Mikhalkin). *The numbers N_d^{trop} of rational tropical plane curves of degree d through $3d-1$ points satisfy the recursion:*

$$N_d^{trop} = \sum_{\substack{d_1+d_2=d \\ d_1\geq 1, d_2\geq 1}} \left[\binom{3d-4}{3d_1-2} d_1^2 d_2^2 - \binom{3d-4}{3d_1-1} d_1^3 d_2\right] N_{d_1}^{trop} N_{d_2}^{trop}. \quad (65)$$

I first saw this formula when I was an undergraduate and I thought it was ghastly! Now I look at it and it looks remarkably simple and elegant. This is the same recursion satisfied by the number of complex algebraic plane rational curves passing through $3d-1$ points. Since we have checked that $N_1 = N_1^{trop} = 1$ (there is exactly one (tropical) line through two points in the plane), we get the correspondence theorem as a simple corollary.

COROLLARY 6.1.1. *For any $d \geq 1$,*

$$N_d = N_d^{trop}. \quad (66)$$

The strategy of proof is the following. We are going to write down a map of constant degree $\Pi : M_{0,3d}^{trop}(\mathbb{P}^2, d) \to \mathbb{R}^{6d-2} \times M_{0,4}^{trop}$. Then we fix a generic point of $\mathbb{R}^{6d-2}$ and consider two points Γ_1, Γ_2 on different ends of $M_{0,4}^{trop}$: their inverse images, counted with multiplicities, must give the degree of Π, so in particular must be equal. Formula (65) then falls out from an explicit analysis of the maps in the two inverse images, and their multiplicities.

STEP I: CONSTRUCTION OF Π.

Let $d \geq 2$ and $n = 3d$, and consider the morphism:

$$\Pi : M_{0,n}^{trop}(\mathbb{P}^2, d) \to \mathbb{R} \times \mathbb{R} \times \underbrace{\mathbb{R}^2 \times \ldots \times \mathbb{R}^2}_{(n-2)\ copies} \times M_{0,4}^{trop} \quad (67)$$

defined as:

$$\Pi = ev_{1,x} \times ev_{2,y} \times ev_3 \times \ldots \times ev_n \times f_4, \quad (68)$$

where:

- $ev_{1,x}$ is the function that evaluates only the first coordinate of the first mark.

FIGURE 6.1. The curves $\gamma_1, \gamma_2 \in M_{0,4}^{trop}$.

- $ev_{2,y}$ is the function that evaluates only the second coordinate of the second mark.
- for $3 \leq i \leq n$, ev_i is the ordinary evaluation function, evaluating both coordinates of the i-th mark.
- f_4 is the forgetful morphism that forgets all marks $i \geq 5$, as well as the map φ.

QUESTIONS/ACTIVITIES 6.1.

(1) Compute $\Pi(\Gamma, f)$ for the tropical stable map in Exercise 5.2.
(2) Show that the dimension of both domain and codomain of Π is $6d - 1$.
(3) Prove that the degree of Π is constant.
(4) Fix $\gamma \in M_{0,4}^{trop}$, $x_0, y_0 \in \mathbb{R}$, points $P_3, \ldots, P_n \in \mathbb{R}^2$ and consider the inverse image
$$\Pi^{-1}(\{x_0\} \times \{y_0\} \times \{P_3\} \times \ldots \times \{P_n\} \times \gamma). \tag{69}$$
Give a geometric description of the points in this inverse image.

STEP II: THE DEGREE OF Π.

The degree of Π is computed by fixing a (general) point in the codomain and counting the inverse images with appropriate multiplicities (as in Definition 3.7).

This will be a fairly long and technical computation, so we break it up into more sub-steps.

STEP II.A: TWO POINTS ON $M_{0,4}^{trop}$.

Let $\gamma_1 \in M_{0,4}^{trop}$ be a four pointed tropical rational curve where the first and second mark are on one vertex, the third and fourth on the other vertex, and the compact edge is very long. Let $\gamma_2 \in M_{0,4}^{trop}$ be a four pointed tropical rational curve where the first and third mark are on one vertex, the second and fourth on the other vertex, and the compact edge is very long. See Figure 6.1.

"Very long" doesn't seem a particularly precise mathematical concept. The following questions will help us make it so.

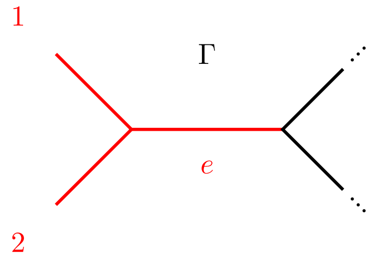


FIGURE 6.2. A part of the source curve Γ for a tropical stable map $(\Gamma, \varphi) \in \Pi^{-1}(x_0, y_0, P_3, \dots P_n, \gamma_1)$, where the edge e which is contracted by φ is adjacent to the first two marks. The dots signify that the curve Γ continues.

QUESTIONS/ACTIVITIES 6.2.

(1) Remember we have fixed $x_0, y_0 \in \mathbb{R}$, and points $P_3, \dots, P_n \in \mathbb{R}^2$. Argue that there exist a uniform bound for the length of the compact egde of any curve $F(\Gamma, \varphi), \Gamma \in \mathcal{C}$ **unless** the map $\varphi : \Gamma \to \mathbb{R}^2$ contracts some edge.
(2) Argue that if one chooses the length of the compact edges of Γ_1, Γ_2 large enough, then all maps in the inverse images of Γ_1, Γ_2 must contract (at least) one edge.

STEP II.B: THE MULTIPLICITY OF $(\Gamma, \varphi) \in \Pi^{-1}(x_0, y_0, P_3, \dots P_n, \gamma_1)$.

Let (Γ, φ) be a tropical map in the inverse image $\Pi^{-1}(\gamma_1)$. We want to compute $mult_\Pi(\Gamma, \varphi)$. We know that Γ must have an edge contracted by φ, and such edge separates the marks $1, 2$ from the marks $3, 4$.

We study separately two cases. First, when the contracted edge is adjacent to the marks 1 and 2.

QUESTIONS/ACTIVITIES 6.3. Let $(\Gamma, \varphi) \in \Pi^{-1}(x_0, y_0, P_3, \dots P_n, \gamma_1)$ be such that the contracted edge e is adjacent to the marks $1, 2$, as in Figure 6.2.

(1) Where does the edge e map in the plane?
(2) Replace the tripod consisting of the marks $1, 2$ and the edge e with a unique end marked by the symbol 0, that contracts to the image point of e. This construction gives a point $(\tilde{\Gamma}, \tilde{\varphi}) \in M^{trop}_{0,3d-1}(\mathbb{P}^{2,trop}, d)$.
(3) Denote by P_0 the point (x_0, y_0). Then note that $(\tilde{\Gamma}, \tilde{\varphi})$ is a curve contributing to the count of rational curves of degree d through $3d - 1$ points.
(4) Viceversa, for any $(\tilde{\Gamma}, \tilde{\varphi})$ contributing to the count of rational curves of degree d through $3d-1$ points, show that by replacing the mark 0

with a tripod, one obtains a point $(\Gamma, \varphi) \in \Pi^{-1}(x_0, y_0, P_3, \dots P_n, \gamma_1)$ such that the contracted edge e is adjacent to the marks $1, 2$.

(5) Show (writing as little as possible) that the matrix that computes $mult_{\Pi}(\Gamma, \varphi)$ and the matrix that computes the contribution of $(\tilde{\Gamma}, \tilde{\varphi})$ to the count N_d^{trop} have the same determinant.

(6) Conclude that that the contribution to the weighted count of points in $\Pi^{-1}(x_0, y_0, P_3, \dots P_n, \gamma_1)$ by curves where the contracted edge is adjacent to the marks $1, 2$ equals N_d.

The next case is when the contracted edge e is not adjacent to the marks $1, 2$ in Γ, but somewhere in the middle of the curve Γ.

LEMMA 6.1. *Let* $(\Gamma, \varphi) \in \Pi^{-1}(x_0, y_0, P_3, \dots P_n, \gamma_1)$ *with non-zero contribution to* $deg(\Pi)$ *be such that the contracted edge* e *is not adjacent to the marks* $1, 2$.

(1) *By cutting the edge* e *one obtains two maps* $(\Gamma_1, \varphi_1), (\Gamma_2, \varphi_2)$ *of degree* d_1, d_2 *with* $d_1 + d_2 = d$.
(2) *The curve* Γ_1 *contains the marks* $1, 2$ *and* $3d_1 - 1$ *further markings.*
(3) *The curve* Γ_2 *contains the marks* $3, 4$ *and* $3d_2 - 3$ *further markings.*
(4) *Denote by* L_1 *the line* $x = x_0$ *and by* L_2 *the line* $y = y_0$. *Also, to avoid excessive notation, denote by* Γ_i *both the source of the map and the image plane tropical curve. We have:*

$$mult_{(\Gamma,\varphi)}(\Pi) = mult_{(\Gamma_1,\varphi_1)}(Ev) \cdot mult_{(\Gamma_2,\varphi_2)}(Ev) \cdot mult_{\varphi(e)}(\Gamma_1, \Gamma_2) \cdot \\ mult_{\varphi(1)}(\Gamma_1, L_1) \cdot mult_{\varphi(2)}(\Gamma_2, L_2). \tag{70}$$

PROOF. The first statement follows from the balancing condition: when cutting the edge e, Γ breaks into two connected components, and the sum of the direction vectors of the ends for each of the components must be zero. Therefore each component must contain the same number of down, left and diagonal ends, showing that the restriction of φ to each component is a tropical map to the tropical projective plane.

For the second and third statements, since the edge e separates the marks $1, 2$ and $3, 4$ we are just choosing to call Γ_1 the connected component containing the first two marks and Γ_2 the connected component containing the third and fourth. For the second part of these statements, notice that since the first two marked points are required to map to lines, this does not put any restriction on the map (Γ_1, φ_1). Therefore we may impose that a map of degree d_1 passes through at most $3d_1 - 1$ marks. On the other hand, the third and fourth marked points are required to map to fixed points P_3, P_4, and therefore we may impose that a map of degree d_2 passes through at most $3d_2 - 3$ further points. But since the total of marked points of Γ is $3d$, the maximum constraint must be attained by each component.

Now we come to the tricky part, where we must carefully analyze the multiplicities.

First, we observe (See Exercise 5.3) that the choice of using the first vertex for fixing the translation factor in the bijection $M_{0,3d}^{trop}(\mathbb{P}^2, d) \to M_{0,6d}^{trop} \times \mathbb{R}^2$ is irrelevant: choosing any other vertex does not alter the determinant of

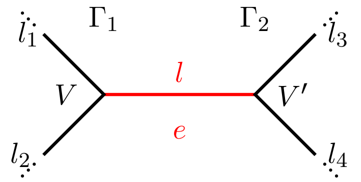


FIGURE 6.3. A part of the source curve Γ for a tropical stable map $(\Gamma, \varphi) \in \Pi^{-1}(x_0, y_0, P_3, \ldots P_n, \gamma_1)$, where the edge e which is contracted by φ is not adjacent to the first two marks. The edges of lengths l_1 and l_2, oriented outwards from V, are mapped via φ with direction vectors $\pm\mathbf{v}$. The edges of lengths l_3, l_4, are mapped via φ with direction vectors $\pm\mathbf{w}$.

the matrix M. We therefore choose to evaluate the vertex V adjacent to the edge e and belonging to Γ_1.

Figure 6.3 is a local picture of Γ around the contracted edge e. We rearrange rows and columns of M so as to obtain a block decomposed matrix that looks as follows:

	l	lengths in Γ_1	l_1	l_2	l_3	l_4	lengths in Γ_2	$\varphi(V)$
$ev_{1,x}$	0	$*$	v_x	0	0	0	0	1 0
$ev_{2,y}$	0	$*$	v_y	0	0	0	0	0 1
evaluations of coordinates of points behind l_1	0	$*$	$\mathbf{v}$	0	0	0	0	$\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}$
evaluations of coordinates of points behind l_2	0	$*$	0	$-\mathbf{v}$	0	0	0	$\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}$
evaluations of coordinates of points behind l_3	0	0	0	0	$\mathbf{w}$	0	$*$	$\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}$
evaluations of coordinates of points behind l_4	0	0	0	0	0	$-\mathbf{w}$	$*$	$\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}$
f_4	1	$*$	$*$	$*$	$*$	$*$	$*$	$*$

Here is how to interpret this matrix: we have no control of the blocks filled with $*$'s. For each marked point, the corresponding ev_i gives two rows of the matrix, to be chosen among the four depicted in the matrix based on whether the i-th mark lives in Γ_1 or Γ_2, and whether it preceeds or follows the edge e. We are assuming here that the marks 1 and 2 are both behind l_1. If one follows l_1 and the other l_2, the matrix needs to be slightly changed but the proof will follow the same way.

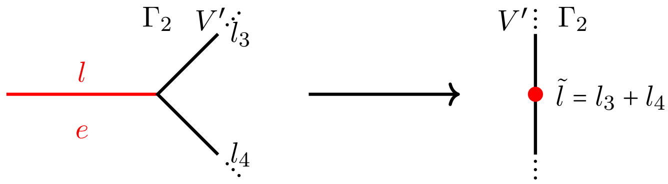


FIGURE 6.4. The tropical curve Γ_2 does not have the edge e. Therefore the two adjacent edges are replaced by a unique edge of length $\tilde{l} = l_3 + l_4$.

This is a $6d - 1 \times 6d - 1$ matrix. For the purposes of calculating the determinant we can remove the first column and the last row. Now we perform the following column operations that do not change the absolute value of the determinant:

- Move the last two columns to the front;
- Multiply the first column of $\varphi(V)$ by v_x, the second column by v_y and add them to the column l_2; subtract the column l_1;
- Subtract l_4 from l_3;
- Since $\mathbf{v}, \mathbf{w}$ are linearly independent, there exists a linear combination $c_1(\mathbf{v}, \mathbf{w})$ that equals the first standard basis vector $\mathbf{e_1}$, and another linear combination $c_2(\mathbf{v}, \mathbf{w})$ that equals $\mathbf{e_2}$. Subtract $c_1(\tilde{l}_2, l_4 - l_3)$ from the first column and $c_2(\tilde{l}_2, l_4 - l_3)$ from the second.

The resulting matrix looks like this:

	$\varphi(V) - C(\tilde{l}_2, l_4 - l_3))$	lengths in Γ_1	l_1	$\tilde{l}_2 = l_2 - l_1 + \varphi(V)\mathbf{v}$	$l_3 - l_4$	l_4	lengths in Γ_2
$ev_{1,x}$	1 0	$*$	v_x	0	0	0	0
$ev_{2,y}$	0 1	$*$	v_y	0	0	0	0
evaluations of coordinates of points behind l_1	$\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}$	$*$	$\mathbf{v}$	0	0	0	0
evaluations of coordinates of points behind l_2	$\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}$	$*$	0	0	0	0	0
evaluations of coordinates of points behind l_3	0	0	0	$\mathbf{v}$	$\mathbf{w}$	0	$*$
evaluations of coordinates of points behind l_4	0	0	0	$\mathbf{v}$	$\mathbf{w}$	$-\mathbf{w}$	$*$

We now have a block decomposition of the matrix. Let us first focus on the lower block M_{SE}, and observe that it is very similar to the matrix A_2 computing the multiplicity of Ev for the curve Γ_2.

Referring to Figure 6.4 for notation:

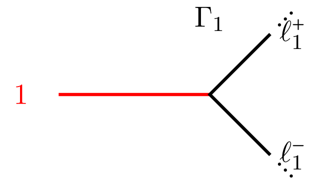


FIGURE 6.5. The portion of tropical curve Γ_1 near the first mark.

$A_2 =$

	$\varphi(V')$	$\tilde{l}$	lengths in Γ_2
evaluations of coordinates of points before V'	$\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}$	0	$*$
evaluations of coordinates of points after V'	$\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}$	$-\mathbf{w}$	$*$

Precisely, let $M_{\mathbf{v},\mathbf{w}}$ be a block matrix consisting of a 2×2 upper block containing the coordinates of the vectors $\mathbf{v}, \mathbf{w}$ and a lower identity block. Then we have:

$$M = A_2 \cdot M_{\mathbf{v},\mathbf{w}} \tag{71}$$

and therefore

$$det(M) = det(A_2)det(M_{\mathbf{v},\mathbf{w}}) = mult_{(\Gamma_2,\varphi_2)}(Ev) \cdot mult_{\varphi(e)}(\Gamma_1, \Gamma_2). \tag{72}$$

We now turn to the north-west block of M, which we denote M_{NW}. This is a $6d_1 \times 6d_1$ matrix and it is quite similar to the matrix A_1 computing the multiplicity of Ev for Γ_1: the problem is that A_1 is a square matrix of size $6d_1 - 2$. The issue arises with the fact that the marks 1 and 2, which are mapping to L_1 and L_2, are subdividing two edges of Γ, and A_1 does not see this subdivision. Refer to Figure 6.5 Denote by $\mathbf{s}$ the direction vector of the edge of Γ_1 containing the first mark, and denote ℓ_1^+, ℓ_1^- the lengths of the two adjacent edges. Then we have:

$M_{NW} =$

	$\varphi(V)$	ℓ_1^-	ℓ_1^+	other lengths in Γ_1	l_1
$ev_{1,x}$	$1 \quad 0$	s_x	0	$*$	v_x
$ev_{2,y}$	$0 \quad 1$	$\star$	$\star$	$*$	v_y
evaluations of coordinates of points before 1	$\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}$	0	0	$*$	$\mathbf{v}$
evaluations of coordinates of points after 1	$\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}$	$\mathbf{s}$	$\mathbf{s}$	$*$	$\mathbf{v}$

Subtracting ℓ_1^+ from ℓ_1^- we get:

	$\varphi(V)$	$\ell_1^- - \ell_1^+$	ℓ_1^+	other lengths in Γ_1	l_1
$ev_{1,x}$	1 0	s_x	0	$*$	v_x
$ev_{2,y}$	0 1	0	$\star$	$*$	v_y
evaluations of coordinates of points before 1	$\begin{matrix}1 & 0\\ 0 & 1\end{matrix}$	0	0	$*$	$\mathbf{v}$
evaluations of coordinates of points after 1	$\begin{matrix}1 & 0\\ 0 & 1\end{matrix}$	0	$\mathbf{s}$	$*$	$\mathbf{v}$

,

which tells us that the absolute value of the determinant of M_{NW} equals $|s_x|$ times the determinant of the matrix obtained by removing the row corresponding to $ev_{1,x}$ and the column $\ell_1^- - \ell_1^+$. We note that $|s_x| = mult_{\varphi(1)}(\Gamma_1, L_1)$. A similar argument applies for the the row $ev_{2,y}$, and the remaining matrix is precisely A_1. Therefore we have:

$$det(M_{NW}) = mult_{(\Gamma_2,\varphi_2)}(Ev) \cdot mult_{\varphi(1)}(\Gamma_1, L_1) \cdot mult_{\varphi(2)}(\Gamma_1, L_2). \quad (73)$$

The lemma is now proven by combining (72) and (73). □

STEP II.C: ADDING ALL THE MULTIPLICITIES IN $\Pi^{-1}(x_0, y_0, P_3, \dots P_n, \gamma_1)$

Choose $3d_1 - 1$ points among $P_3, \dots, P_n$ and call them $Q_1, \dots Q_{3d_1-1}$ (call $R_1, \dots R_{3d_2-3}$ the complementary points); consider the two following sets:

- $X_1 = \{(\Gamma_1, \varphi_1) \in M^{trop}_{0,3d_1+1}(\mathbb{P}^2, d_1) | \varphi_1(i) = Q_i, \varphi_1(3d_1) = L_1, \varphi(3d_1 + 1) = L_2\}$;
- $X_2 = \{(\Gamma_2, \varphi_2) \in M^{trop}_{0,3d_2-1}(\mathbb{P}^2, d_1) | \varphi_1(i) = R_i, \varphi_1(3d_2 - 2) = P_1, \varphi(3d_2 - 1) = P_2\}$.

We compute the weighted sums:

$$\sum_{X_2} mult_{(\Gamma_2,\varphi_2)}(Ev) = N_{d_2}, \quad (74)$$

and

$$\sum_{X_1} mult_{(\Gamma_1,\varphi_1)}(Ev) \cdot mult_{\varphi_1(3d_1)}(\Gamma_1, L_1) \cdot mult_{\varphi_1(3d_1+1)}(\Gamma_1, L_2) = d_1^2 N_{d_1}, \quad (75)$$

where the factor d_1^2 comes from Bézout's theorem. Note that we can construct a correspondence:

$$\iota : X_1 \times X_2 \to \Pi^{-1}(x_0, y_0, P_3, \dots P_n, \gamma_1) \quad (76)$$

by associating to a pair $(\Gamma_1, \varphi_1), (\Gamma_2, \varphi_2)$ all the curves in $\Pi^{-1}(x_0, y_0, P_3, \dots P_n, \gamma_1)$ obtained by inserting a contracting edge of appropriate length at any of the points x of intersection of Γ_1 and Γ_2. and that the contribution to the degree of Π by elements in the image of ι is:

$$\sum_{Im(\iota)} mult_{(\Gamma_1,\varphi_1)}(Ev) \cdot mult_{(\Gamma_2,\varphi_2)}(Ev) \cdot mult_{\varphi_1(3d_1)}(\Gamma_1, L_1) \cdot$$
$$mult_{\varphi_1(3d_1+1)}(\Gamma_1, L_2) \cdot mult_x(\Gamma_1, \Gamma_2) = d_1^3 d_2 N_{d_1} N_{d_2}, \quad (77)$$

We finally note that there are $\binom{3d-4}{3d_1-1}$ ways to choose $3d_1 - 1$ points among $P_3, \ldots, P_n$. The images of the correspondences ι for any two of these choices are disjoint, and the union of such images exausts all elements in $\Pi^{-1}(x_0, y_0, P_3, \ldots P_n, \gamma_1)$ such that, after cutting the contracted edge, one is left with a pair of maps of degrees d_1 and d_2.

Adding over all pairs of degrees $d_1 + d_2 = d$, and remembering the contribution from when the contracted edge was adjacent to both marks $1, 2$, we obtain

$$deg(\Pi) = N_d + \sum_{\substack{d_1 + d_2 = d \\ d_1 \geq 1, d_2 \geq 1}} \left[\binom{3d-4}{3d_1-1} d_1^3 d_2\right] N_{d_1}^{trop} N_{d_2}^{trop}. \quad (78)$$

Questions/Activities 6.4.

(1) Compute $deg(\Pi)$ using $\Pi^{-1}(x_0, y_0, P_3, \ldots P_n, \gamma_2)$.
(2) Obtain a proof of Theorem 6.1 by setting the two computations of $deg(\Pi)$ equal to each other.
(3) Pat yourself on the back and highfive your neighbor for making it through these intense 12 hours of math! Hopefully you learned a lot and had fun in the process!