Exercise Sheet 3
Introduction to Commutative Algebra and Algebraic Geometry

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Exercise 1

Let $X_1 = V(x^3 - 3x + 2 - y^2), \quad X_2 = V(x-1)$ be two varieties in $\mathbb{C}^2$. Make a sketch of the real part of $X_1, X_2 \subset \mathbb{R}^2$ in the same coordinate system and compute

$$\sqrt{I(X_1) + I(X_2)} \subset \mathbb{C}[X,Y].$$

Exercise 2

Let K be an algebraically closed field. Consider the variety $V = V(Y^2 - X^3)$ of the cuspidal cubic $Y^2 - X^3 \in K[X,Y]$. Let $K[V]$ be the coordinate ring of V. Prove

$$K[V] \cong K[T^2, T^3].$$

Exercise 3

Let K be an algebraically closed field. Find all polynomials $f \in K[X_1, \dots, X_n]$ for which

$$V(X_{n+1}^2 - f) \subset K^{n+1}$$

is irreducible. Prove your claim.

Exercise 4

Let K be a field and let

$$0 \to V_0 \to \cdots \to V_n \to 0$$

be an exact sequence of finite dimensional vector spaces over K. Prove:

$$\sum_{i=0}^{n} (-1)^i \dim(V_i) = 0.$$

Submission: Work in groups of up to three students. Submit your solutions either by uploading them to URM or by placing them in the mailbox of Parisa Ebrahimian or Veronika Körber (Room A16, C-Building) by Tuesday, 4 Nov 2025.