# Exercise Sheet 6
## Introduction to Commutative Algebra and Algebraic Geometry

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**Wintersemester 2025/2026** **Assistants:** Parisa Ebrahimian and Veronika Körber

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## Exercise 1 (Product criterion).

Let $K$ be a field, $>$ be a monomial order, $f, g \in K[\underline{x}]$, $\gcd(\mathrm{LM}(f), \mathrm{LM}(g)) = 1$.
Show that there is a polynomial division with remainder of $\mathrm{spoly}(f, g)$ by $(f, g)$ with remainder 0. *Hint:* Show first that

$$\mathrm{spoly}(f, g) = a_0 f + b_0 g$$

for $a_0 = -\mathrm{tail}(g)$ and $b_0 = \mathrm{tail}(f)$, and then define recursively $a_i = \mathrm{tail}(a_{i-1})$ and $b_i = \mathrm{tail}(b_{i-1})$. Consider the maximal value $N$ such that

$$u \cdot \mathrm{spoly}(f, g) = a_N f + b_N g$$

for some element $u \in K[\underline{x}]^*$, and distinguish the two cases that $\mathrm{LT}(a_N f) + \mathrm{LT}(b_N g)$ vanishes or does not vanish.

## Exercise 2.

The degree lexicographical ordering $>_{Dp}$ on $\mathrm{Mon}_n$ is defined by

$$x^\alpha >_{Dp} x^\beta :\Leftrightarrow |\alpha| > |\beta| \text{ or } (|\alpha| = |\beta| \text{ and } \exists k : \alpha_1 = \beta_1, \dots, \alpha_{k-1} = \beta_{k-1}, \alpha_k > \beta_k).$$

A polynomial

$$f = \sum_{\alpha \in \mathbb{N}^n} a_\alpha \underline{x}^\alpha \in K[x_1, \dots, x_n]$$

is called *homogeneous* if for all $\alpha$ with $a_\alpha \neq 0$ the absolute value $|\alpha|$ is constant.

Show that a monomial ordering $>$ on $\mathrm{Mon}_n$ equals $>_{Dp}$ if and only if $>$ is a degree ordering and for any homogeneous $f \in K[\underline{x}]$ with $\mathrm{LM}(f) \in K[x_k, \dots, x_n]$, we have $f \in K[x_k, \dots, x_n]$, $k = 1, \dots, n$.

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**Submission:** Work in groups of up to three students. Submit your solutions either by uploading them to URM or by placing them in the mailbox of Parisa Ebrahimian or Veronika Körber (Room A16, C-Building) by Tuesday, 25 Nov 2025

## Exercise 3.

Apply IDBuchberger to the following triple $(g, G, >)$:

$$g = x^4 + y^4 + z^4 + xyz, \quad G = \left\{ \frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z} \right\}, \quad >_{dp} .$$