Exercise Sheet
Introduction to Commutative Algebra and Algebraic Geometry

Eberhard-Karls-Universität Tübingen **Profesor:** Hannah Markwig
Wintersemester 2025/2026 **Assistants:** Parisa Ebrahimian and Veronika Körber

---

You can use any computer algebra system you want for these exercises.

Exercise 1.

Let $f, g \in K[x]$, the polynomial ring in *one* variable. Express the greatest common divisor and the least common multiple of f and g in terms of elements in $\operatorname{syz}(f, g)$ and derive an algorithm to compute these, assuming we can compute a Gröbner basis of $\operatorname{syz}(f, g)$.

Exercise 2.

Let

$$V = V\big((x+y)\cdot(x-y)\cdot(x+z^2)\big) \subset \mathbb{C}^3$$

and

$$W = V\big((x+z^2)\cdot(x-y)\cdot(z+y)\big) \subset \mathbb{C}^3.$$

Draw sketches of V and W in $\mathbb{R}^3$, and try to guess the ideal of $\overline{V \setminus W}$. How could you verify your guess? Answer with an appropriate algorithm and describe the steps accurately without doing all the computations.

Exercise 3.

(a) Is the polynomial $f = x^7 + x^2$ in the radical of $I = \langle x^{11} + x^6,\ yx^4 + x^8,\ y^3 + x^2\rangle$? Use Algorithm 2.5.5 to prove your claim.

Hint: Eliminate y first.

(b) Let

$$I = \langle x^2 + 2y^2 - 3,\ x^2 + xy + y^2 - 3\rangle \subset \mathbb{Q}[x, y].$$

Compute $I \cap \mathbb{Q}[y]$.

Submission: Work in groups of up to three students. Submit your solutions either by uploading them to URM or by placing them in the mailbox of Parisa Ebrahimian or Veronika Körber (Room A16, C-Building) by Tuesday, 9 Dec 2025