Exercise Sheet 1
Introduction to Commutative Algebra and Algebraic Geometry

Eberhard-Karls-Universität Tübingen **Profesor:** Hannah Markwig
Wintersemester 2025/2026 **Assistant:** Parisa Ebrahimian and Veronika Körber

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Exercise 1

Let R be a commutative ring with one. For a subset $S \subset R$ the ideal generated by S is defined as the smallest ideal of R containing S:

$$\langle S \rangle := \bigcap_{J \text{ ideal of } R, S \subset J} J.$$

Show:

$$\langle S \rangle = \left\{ \sum_{i=1}^{n} \lambda_i s_i \mid n \in \mathbb{N}, \lambda_i \in R, s_i \in S \ \forall i \in \{1, \ldots, n\} \right\}.$$

Exercise 2

Let K be a field and let $R = K[x]$. Let $I = aR$, $J = bR \subset K[x]$ be two principal ideals. Show

$$I + J = \langle \gcd(a, b) \rangle,$$
$$I \cap J = \langle \operatorname{lcm}(a, b) \rangle,$$

where gcd denotes the greatest common divisor and lcm the least common multiple.

Exercise 3

Let R be a commutative ring with one. Let $I \subset R$ be an ideal. Show that $\sqrt{I}$ is an ideal in R.

Exercise 4

Let $R = \mathbb{C}[x, y]$.

Submission: In groups of up to three students. Submit by placing solutions in Parisa Ebrahimian's mailbox (Raum A16, C-Bau) by Tuesday, 21 October 2025.

- Compute $\langle x^2, y\rangle \cap \langle x-1, y-1\rangle$, and use this to show:

$$V(\langle x^2, y\rangle \cap \langle x-1, y-1\rangle) = V(x^2, y) \cup V(x-1, y-1).$$

Hint: Show that the two ideals are coprime and use the Chinese Remainder Theorem.

- Make a sketch of the real part of the variety of $I_1 = \langle y - x^2 + x + 4\rangle$ and of the variety of $I_2 = \langle x - 2\rangle$. Then compute $V(I_1 + I_2)$ and verify that $V(I_1 + I_2)$ is contained in the varieties of I_1 and I_2. Repeat this for $I_1 = \langle y - x^3 + 3x^2 + x - 3\rangle$, $I_2 = \langle -y - x + 3\rangle$.

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