Exercise Sheet 11
Introduction to Commutative Algebra and Algebraic Geometry

Eberhard-Karls-Universität Tübingen
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Exercise 1.

For a ring extension $R' \supset R$, we say $\alpha \in R'$ is *integral* over R if there exists $f \in R[x] \setminus \{0\}$ with $f(\alpha) = 0$ and $\mathrm{LC}(f) = 1$. R' is called integral over R if $\forall \alpha \in R'$: α is integral over R.

Let K be a field. Let R be a finitely generated K-algebra without zero divisors and let $\beta_1, \ldots, \beta_d \in R \subset \mathrm{Quot}(R)$. Prove: If R is integral over $K(\beta_1, \ldots, \beta_d)$, then $\mathrm{Quot}(R)$ is algebraic over $K(\beta_1, \ldots, \beta_d)$.

Exercise 2.

(Semicontinuity of the dimension) Let X be an affine variety with irreducible components $X_1, \ldots, X_r$. For $x \in X$ we set

$$\dim(X_x) := \max\{\dim(X_i) : 1 \leq i \leq r,\ x \in X_i\}.$$

Prove: The function

$$X \to \mathbb{Z}, \qquad x \mapsto \dim(X_x)$$

is upper semicontinuous, i.e., $\forall x \in X$ there is an open neighbourhood $U \subset X$ with $\dim(X_u) \leq \dim(X_x)$ for all $u \in U$.

Exercise 3.

Let K be an algebraically closed field.

(a) Determine the field of rational functions and the dimension of the cuspidal cubic

$$X = V(x^2 - y^3) \subset K^2.$$

(b) Let U be a d-dimensional vector subspace of K^n. Determine the dimension of U as an affine variety in K^n and compare it to the dimension of U as a K-vector space.

Submission: Work in groups of up to three students. Submit your solutions either by uploading them to URM or by placing them in the mailbox of Parisa Ebrahimian (Room A16, C-Building) by Tuesday, 20 Jan, 2026.