

Exercise Sheet 12

Introduction to Commutative Algebra and Algebraic Geometry

Eberhard-Karls-Universität Tübingen
Wintersemester 2025/2026

Profesor: Hannah Markwig
Assistant: Parisa Ebrahimian

Exercise 1.

Let K be an algebraically closed field, $f, g \in K[x_1, \dots, x_n]$, $V(f, g) \subset K^n$. Prove the equivalence of the following statements:

- (i) $\gcd(f, g) = 1$.
- (ii) $\dim(V(f, g)) \leq n - 2$.

Exercise 2.

Show that the statement from Theorem 4.2.1 (2) does not generally hold if K is not algebraically closed. For this consider $K = \mathbb{R}$ and $X = Y = K$ and

$$\varphi : X \rightarrow Y, \quad t \mapsto t^2$$

and the ideal $J = \langle x^2 + 1 \rangle \subset K[x]$.

Exercise 3.

Determine the number of irreducible components, the dimension, and the ring of global functions for every fibre of the following morphisms:

- (i) $\varphi : K^2 \rightarrow K, (z, w) \mapsto zw$,
- (ii) $\psi : K^2 \rightarrow K^2, (z, w) \mapsto (zw, w)$.