

Exercise Sheet 2

Introduction to Commutative Algebra and Algebraic Geometry

Eberhard-Karls-Universität Tübingen
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Professor: Hannah Markwig
Assistant: Parisa Ebrahimian and Veronika Körber

Exercise 1

Let K be a field. Consider $R := K[T^2, T^3] \subset K[T]$ and prove the following statements:

1. R is Noetherian.
2. $\sqrt{\langle T^2 \rangle} = \langle T^2, T^3 \rangle$.

Exercise 2

Let (X, Ω) be a topological space. The closure of a subset $A \subset X$ is the intersection $\overline{A}$ of all closed subsets $B \subset X$ with $A \subset B$. Show:

1. A subset $A \subset X$ is closed in X if and only if $A = \overline{A}$.
2. For every finite union $A := A_1 \cup \dots \cup A_n$ of subsets $A_1, \dots, A_n \subset X$ the following applies:

$$\overline{A} = \overline{A_1} \cup \dots \cup \overline{A_n}.$$

3. Let $A \subset B \subset X$ be subsets. The closure of A in B with respect to the subspace topology is given by $\overline{A} \cap B$.

The subspace topology of $B \subset X$ is given by the following system of open sets:

$$\Omega_B = \{Y \cap B \mid Y \subset X \text{ open}\}.$$

Exercise 3

1. Let $V_1 = \{1, 3, 5\} \subset \mathbb{R}$ and $V_2 = \{1, 2, 3, 4, 5\} \subset \mathbb{R}$. Compute $I(V_1) \subset \mathbb{R}[x]$ and $I(V_2) \subset \mathbb{R}[x]$. Prove $I(V_2) \subset I(V_1)$.
2. Let $V_3 = \{(0, 0), (0, 2), (1, 0), (1, 1)\}$, $V_4 = \{(1, -1), (1, 0), (0, 0), (3, 1), (0, 2)\} \subset \mathbb{R}^2$. Compute $I(V_3)$, $I(V_4) \subset \mathbb{R}[x, y]$ and $I(V_3 \cap V_4)$.
3. Let $M \neq \emptyset$ be an arbitrary finite subset of $\mathbb{R}^2$. Find a polynomial $f \in \mathbb{R}[x, y]$ such that $M = V(f)$.
4. Consider $\mathbb{Z} \subset \mathbb{R}$. Compute $I(\mathbb{Z}) \subset \mathbb{R}[x]$.

Submission: In groups of up to three students. Submit by placing solutions in Parisa Ebrahimian's mailbox (Raum A16, C-Bau) by Tuesday, 28 October 2025.

Exercise 4

Let A, B, C be finite groups and let

$$0 \rightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \rightarrow 0$$

be a short exact sequence of group homomorphisms, i.e., α is injective, β is surjective, and $\ker(\beta) = \operatorname{im}(\alpha)$. Prove:

$$|B| = |A| \cdot |C|.$$