Exercise Sheet 3
Introduction to Commutative Algebra and Algebraic Geometry

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Exercise 1

Let $X_1 = V(x^3 - 3x + 2 - y^2), \quad X_2 = V(x-1)$ be two varieties in $\mathbb{C}^2$. Make a sketch of the real part of $X_1, X_2 \subset \mathbb{R}^2$ in the same coordinate system and compute

$$\sqrt{I(X_1) + I(X_2)} \subset \mathbb{C}[X, Y].$$

Exercise 2

Let K be an algebraically closed field. Consider the variety $V = V(Y^2 - X^3)$ of the cuspidal cubic $Y^2 - X^3 \in K[X, Y]$. Let $K[V]$ be the coordinate ring of V. Prove

$$K[V] \cong K[T^2, T^3].$$

Exercise 3

Let K be an algebraically closed field. Find all polynomials $f \in K[X_1, \ldots, X_n]$ for which

$$V(X_{n+1}^2 - f) \subset K^{n+1}$$

is irreducible. Prove your claim.

Exercise 4

Let K be a field and let

$$0 \to V_0 \to \cdots \to V_n \to 0$$

be an exact sequence of finite dimensional vector spaces over K. Prove:

$$\sum_{i=0}^{n} (-1)^i \dim(V_i) = 0.$$

Submission: In groups of up to three students. Submit by placing solutions in Parisa Ebrahimian's mailbox (Raum A16, C-Bau) by Tuesday, 4 Nov 2025.