Exercise Sheet 4
Introduction to Commutative Algebra and Algebraic Geometry

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Exercise 1.

Determine the irreducible components of:

$$V(X^2 - YZ, XZ - X) \subset \mathbb{C}^3.$$

Exercise 2.

Let X be a Noetherian topological space and let $X = X_1 \cup \cdots \cup X_n$ be the decomposition into irreducible components. Show: If $U \subset X$ is a non-empty open subset, then the irreducible components of U are exactly the sets

$$X_i \cap U \quad \text{with } i = 1, \ldots, n \text{ for which } X_i \cap U \neq \emptyset.$$

Exercise 3.

(a) Let $\varphi : X \to Y$ be a continuous map of Noetherian topological spaces and let $Z \subset Y$ be the closure of the image $\varphi(X)$. Prove: The maximal number of irreducible components in the minimal decomposition of Z is bounded by the number of irreducible components of the minimal decomposition of X.

(b) Let $X = V(xy + x - y - 1) \subset \mathbb{C}^2$, $Y = \mathbb{C}$ and let $\varphi : X \to Y$ be defined by $\varphi((a, b)) = a$. Prove that X and Y are Noetherian topological spaces, that φ is continuous with respect to the Zariski topology, and that the number of irreducible components of the closure of $\varphi(X)$ is strictly smaller than the number of irreducible components of X.

Exercise 4.

Let K be a field and let $X := \{A \in \mathrm{Mat}(n \times n; K) \mid \mathrm{rank}(A) \leq 1\}$. Prove: X is irreducible in $\mathrm{Mat}(n \times n; K) = K^{n \times n}$.

Submission: In groups of up to three students. Submit by placing solutions in Parisa Ebrahimian's mailbox (Raum A16, C-Bau) by Tuesday, 11 Nov 2025.