

INTRODUCTION TO SCHEMES

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ABSTRACT. Some notes for the course *Introduction to Schemes* taught at the University of Tübingen in the Winter Semester 2026/27.

1. OUTLINE OF THE COURSE

Course description. The course will be an introduction to Grothendieck's *schemes*, which give the basic language of modern algebraic geometry. We will aim to cover schemes, sheaves and cohomology.

Prerequisites. The language of schemes is fairly technical, and the geometric meaning is not immediately clear. That's why previous knowledge of the classical language of algebraic varieties is very useful. One should also know some basic commutative algebra and some basic facts on sheaves.

The minimum prerequisites are in

- H. Markwig, *Introduction to Commutative Algebra and Algebraic Geometry*. Online notes [8].

It would be even better to know the material from

- A. Gathmann, *Algebraic Geometry*. Online notes, Chapters 1–4 [3].

The best background could be acquired from

- D. Mumford, *The Red Book of Varieties and Schemes*. Springer. Chapter 1 [9].
- R. Hartshorne, *Algebraic Geometry*. Springer. Chapter I [6].

References. We will not follow exactly any particular book, but the main inspirations for the course will be

- A. Gathmann, *Algebraic Geometry*. Online notes. (Gathmann has different versions of these notes on his website, we will follow those from 2002/03.) [3]
- R. Vakil, *The Rising Sea: Foundations of Algebraic Geometry*. Princeton University Press. There is also a *free version*. [11]
- R. Hartshorne, *Algebraic Geometry*. Springer. [6]
- D. Mumford, *The Red Book of Varieties and Schemes*. Springer. [9]

There are many more useful books on this topic. Some of them are:

- Q. Liu, *Algebraic Geometry and Arithmetic Curves*. Oxford University Press. [7]
- U. Görtz and T. Wedhorn, *Algebraic Geometry I: Schemes*. Springer. [4]
- D. Eisenbud and J. Harris, *The Geometry of Schemes*. Springer. [1]
- G. Ellingsrud and J. Ottem, *Introduction to Schemes*. Online notes. [2]

The definitive (and technical) references are:

- A. Grothendieck, *Éléments de géométrie algébrique* (EGA). Publications Mathématiques de l'IHÉS. [5]
- The Stacks Project authors, *The Stacks project*. Online project. [10]

2. WHY SCHEMES?

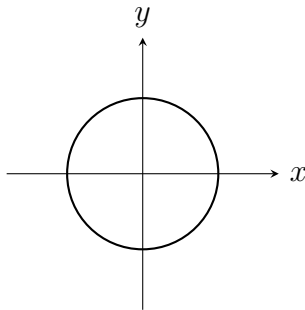
Algebraic geometry is the study of spaces defined by systems of polynomial equations. More precisely, one studies the geometry of these spaces via the algebra of their polynomial equations, and vice versa. *Schemes* are the modern, commonly accepted language for algebraic geometry, introduced by Grothendieck in 1960.

2.1. Affine algebraic sets over algebraically closed fields. The basic objects of algebraic geometry before schemes are *affine algebraic sets* over algebraically closed fields.

Example 2.1. The solutions of $x^2 + y^2 = 1$ in $\mathbf{C}^2$ define an affine algebraic set:

$$V(x^2 + y^2 - 1) = \{(x, y) \in \mathbf{C}^2 \mid x^2 + y^2 - 1 = 0\}.$$

The picture below shows the real part of this set, a circle in $\mathbf{R}^2$.



We recall the more general definition:

Definition 2.2 (Affine algebraic sets). If k is a field, an *affine algebraic set* in k^n is the vanishing locus of a system of polynomials $F_1, \dots, F_s \in k[x_1, \dots, x_n]$, or equivalently of the ideal $I = (F_1, \dots, F_s) \subseteq k[x_1, \dots, x_n]$ that they generate¹:

$$V(I) = \{p \in k^n \mid f(p) = 0 \text{ for all } f \in I\}.$$

Conversely if $X \subseteq k^n$ is an affine algebraic set, its *associated ideal* is the set of all polynomial functions vanishing on it

$$I(X) = \{f \in k[x_1, \dots, x_n] \mid f(p) = 0 \text{ for all } p \in X\}.$$

The affine algebraic sets in k^n are the closed sets of a topology:

Definition 2.3 (Zariski topology). The Zariski topology on k^n has the affine algebraic sets as closed subsets. The Zariski topology on an algebraic set $X \subseteq k^n$ is the induced one.

¹Why is every ideal in $k[x_1, \dots, x_n]$ of this form?

On affine algebraic sets we also have a special class of functions $f: X \rightarrow k$, the *regular functions*. To recall and motivate their definition, we observe that this class of functions should have the following properties:

- As for usual differentiable functions $\mathbf{R} \rightarrow \mathbf{R}$, the property of being regular should be *local*. This means that we should have a notion of a regular function $f: U \rightarrow k$ for any Zariski-open subset $U \subseteq X$. Moreover, the restriction of a regular function to a smaller Zariski-open subset should still be regular and, finally, a function $f: U \rightarrow k$ is regular if and only if there is a cover of $U \subseteq X$ by smaller Zariski-open subsets $U' \subseteq U$ such that $f|_{U'}: U' \rightarrow k$ is regular.
- As for differentiable functions $\mathbf{R} \rightarrow \mathbf{R}$, if $f, g: U \rightarrow k$ are two regular functions on a Zariski-open subset U then their sum $f + g$ and their product $f \cdot g$ are always well defined and should be regular functions on U . Furthermore, if the ratio $\frac{f}{g}$ is well-defined, meaning that g does not vanish at any point of U , then it should be a regular function as well.

These first two properties are independent of the algebro-geometric setting. The third property is where algebraic geometry comes in:

- If $f \in k[x_1, \dots, x_n]$ is a polynomial, then the function $f: X \rightarrow k$ given by the evaluation at the points, should be a regular function.

Putting all these desired properties together we get to the right definition:

Definition 2.4 (Regular function on affine algebraic sets). If $X \subseteq k^n$ is an affine algebraic set and if $U \subseteq X$ is a Zariski-open subset, a regular function on U is a function $f: U \rightarrow k$ such that f is locally the ratio of two polynomials. More precisely, this means that there is a cover of U by open subsets $U' \subseteq U$ such that

$$f|_{U'} = \frac{g|_{U'}}{h|_{U'}} \quad \text{for certain } g, h \in k[x_1, \dots, x_n], \text{ } h \text{ never vanishing on } U'$$

The set of regular functions on U is denoted by

$$\mathcal{O}_X(U) = \mathcal{O}(U) = \{ \text{regular functions } f: U \rightarrow k \}.$$

and by the second property above, it is a ring.

These notions are valid for affine algebraic sets over any field. The key statement that makes everything work so well over algebraically closed fields is *Hilbert's Nullstellensatz*:

Theorem 2.5 (Hilbert's Nullstellensatz - I). *Let k be an algebraically closed field.*

- (1) *If $I \subseteq k[x_1, \dots, x_n]$ is an ideal, then $I(V(I)) = \sqrt{I}$.*
- (2) *There is a bijection*

$$\begin{array}{ccc} \{ \text{affine algebraic sets in } k^n \} & \longleftrightarrow & \{ \text{radical ideals } I \subseteq k[x_1, \dots, x_n] \} \\ X & \longmapsto & I(X) \\ V(I) & \longleftarrow & I \end{array}$$

- (3) *There is a bijection*

$$\begin{array}{ccc} \{ \text{points in } k^n \} & \longleftrightarrow & \{ \text{maximal ideals } \mathfrak{m} \subseteq k[x_1, \dots, x_n] \} \\ (a_1, \dots, a_n) & \longmapsto & (x_1 - a_1, \dots, x_n - a_n) \end{array}$$

- (4) Every regular function on the whole of k^n is polynomial, meaning that the natural map $k[x_1, \dots, x_n] \rightarrow \mathcal{O}(k^n)$ is an isomorphism.

This version of the Nullstellensatz readily implies a version for arbitrary affine algebraic sets over an algebraically closed field.

Theorem 2.6 (Hilbert's Nullstellensatz - II). *Let k be an algebraically closed field and $X \subseteq k^n$ an affine algebraic set.*

- (1) *Every regular function on the whole of X is the restriction of a polynomial function, meaning that the natural map $k[x_1, \dots, x_n]/I(X) \rightarrow \mathcal{O}(X)$ is an isomorphism.*
 (2) *The bijection from Theorem 2.5 induces a bijection*

$$\{\text{affine algebraic sets in } X\} \longleftrightarrow \{\text{radical ideals in } \mathcal{O}(X)\}$$

- (3) *The bijection from Theorem 2.5 induces a bijection*

$$\{\text{points in } X\} \longleftrightarrow \{\text{maximal ideals in } \mathcal{O}(X)\}$$

The Nullstellensatz gives the following guiding principle:

*Over an algebraically closed field, the **geometry** of the affine algebraic set X **corresponds** to the **algebra** of $I(X)$ or of $\mathcal{O}_X(X) = k[x_1, \dots, x_n]/I(X)$.*

However, this setting of affine algebraic sets over algebraically closed fields is too restrictive for many natural mathematical questions. We explain this informally.

2.2. Algebraic sets over fields that are not algebraically closed, and over general rings. What if our field is not algebraically closed? For example, we could be interested, and many mathematicians are, in solutions to polynomial equations over $\mathbf{R}$ or over $\mathbf{Q}$.

Here the previous algebra–geometry correspondence breaks down: for example if $I \subseteq \mathbf{C}[x_1, \dots, x_n]$ is an ideal such that $X = V(I) = \emptyset \subseteq \mathbf{C}^n$, then by the Nullstellensatz $I = \mathbf{C}[x_1, \dots, x_n]$ is the trivial ideal and $\mathcal{O}_X(X) = \mathbf{C}[x_1, \dots, x_n]/I = 0$ is the trivial ring so that trivial geometry means trivial algebra. However, if we consider the field $\mathbf{R}$ and the principal ideal $I = (x^2 + y^2 + 1) \subseteq \mathbf{R}[x, y]$, its algebraic set is

$$V(I) = \{(x, y) \in \mathbf{R}^2 \mid x^2 + y^2 = -1\} = \emptyset,$$

so the geometry of this set is trivial, but the algebra is not: the ideal I is not the whole ring and the ring $\mathbf{R}[x, y]/(x^2 + y^2 + 1)$ is not zero. The algebra knows more than the geometry.

We might even want to look at solutions that are in a ring, for example for Diophantine equations like Fermat's equation

$$\{(x, y, z) \in \mathbf{Z}^3 \mid x^n + y^n = z^n\}, \quad n \geq 1 \text{ a fixed integer.}$$

One option that we might have is to look at the set of solutions of $x^n + y^n = z^n$ inside of an algebraically closed field such as $\mathbf{C}$ and then look at the solutions over $\mathbf{Q}$ or $\mathbf{Z}$ as subsets. However, we might also want to consider the solutions of this equation over a finite field $\mathbf{F}_p$, and these are not a subset of the complex solutions. What we would like

is a geometric object associated to the equation $x^n + y^n = z^n$ that knows its solutions over any ring.

2.3. Beyond algebraic sets: non-reduced rings. Even if we stay over algebraically closed fields, there are issues with the notion of algebraic sets. If we consider the algebraic set given by the single point $0 \in \mathbf{C}$ we have $I(0) = (x) \subseteq \mathbf{C}[x]$, so that

$$\mathbf{C}[x]/(x) \cong \mathcal{O}(\{0\}) = \{\text{regular functions on } \{0\}\}, \quad f \mapsto f(0).$$

We are just taking a polynomial function on $\mathbf{C}$ and evaluating it at zero. However, from analysis we know that it is a good idea to look not only at the value of the function at 0, but also at the higher order approximations of the function. For example, the first order approximations of polynomial functions at 0 are naturally encoded by the ring $\mathbf{C}[x]/(x^2)$:

$$\mathbf{C}[x]/(x^2), \quad f \mapsto f(0) + f'(0)x.$$

It is natural to ask for a geometric space X such that

$$\mathbf{C}[x]/(x^2) \cong \mathcal{O}(X) = \{\text{regular functions on } X\}.$$

This should be something like a “first-order thickening” of the single point 0. However we cannot have anything like this in the usual theory. More abstractly, the rings $\mathcal{O}(X)$ arising from affine algebraic sets are always reduced, while we would want some kind of spaces whose rings of regular functions might also contain nilpotent elements.

2.4. Beyond algebraic sets: a manifold-like approach. Algebraic geometry before schemes used to consider algebraic sets in affine or projective space,

$$\begin{aligned} V(I) \subseteq k^n & \quad I \subseteq k[x_1, \dots, x_n] \text{ an ideal,} \\ V(I) \subseteq \mathbb{P}^n(k) & \quad I \subseteq k[x_0, \dots, x_n] \text{ a homogeneous ideal.} \end{aligned}$$

and these are all embedded objects. However, one of the most important ideas of mathematics in the last 200 years is that we should consider spaces with their own intrinsic geometry, without necessarily seeing them embedded somewhere. This is done in differential geometry via the concept of *smooth manifolds*: those are geometric spaces that locally look like a ball in $\mathbf{R}^n$, and they can be studied locally via analysis. There are also complex manifolds, which are spaces that locally look like a ball in $\mathbf{C}^n$, and they can be studied locally via complex analysis. What we would want in algebraic geometry is then a space that locally looks like some generalization of an algebraic set in a way that takes care of the first two issues, and we would like to study these spaces locally via commutative algebra.

2.5. The language of schemes. The language of schemes gives an answer to all those three points. It generalizes the notion of an affine algebraic set to the notion of an *affine scheme*, which can be defined over an arbitrary ring, and this takes care of the first two points above. Then there is a more general notion of *scheme* which is a geometric space which locally looks like an affine scheme. Finally, since affine schemes are essentially equivalent to commutative rings, schemes can be studied locally via commutative algebra.

3. GEOMETRIC SPACES ARE LOCALLY RINGED SPACES

We begin with an informal discussion about geometric spaces. I want to convince you that a geometric space should be a topological space X together with a certain class of functions.

The structure of a topological space on X gives us some basic notion of continuity on the space X and it is natural. Why the functions? Some intuition comes from science: imagine we live in the topological space X and we want to discover something about its *nature*, or in other words its *geometry*. Science tells us that to understand the nature/geometry of X we should go around, make experiments and measure what comes out. We can think that a certain experiment is a function

$$f: X \rightarrow \mathbf{R} \quad (\text{e.g. "measure temperature"})$$

and that $f(x)$ is the measure that we obtain at the point $x \in X$ (e.g. "temperature at x "). The nature/geometry of X is then encoded in all the possible experiments that we make.

We make two other remarks: the first is that our experiments/functions do not need to be well-defined on the whole of X . For example, if X is the Earth and our experiment/function is "measure the amount of iron in the ground", then this is not defined on the sea. However, if we adopt a "continuous" view of nature, we expect that if we can do an experiment at a point, we should be able to do it close to the point as well. More concretely, we see the experiments as functions defined on the open subsets of X :

$$f: U \rightarrow \mathbf{R}, \quad U \subseteq X \text{ open subset.}$$

The second remark is that if we choose different types of experiments we get a different nature/geometry for the space X . For example, assume that our space X is the Earth and we make different classes of experiments:

- If we measure temperature and atmospheric pressure, we end up with a description of the thermodynamical nature/geometry of the Earth.
- If we measure the chemical composition of the ground, we end up with a description of the geological nature of the Earth.
- If we measure population density and life expectancy we end up with a description of the sociological geometry/nature of the Earth.

In summary, a geometric space should be a topological space X together with a certain class of functions defined on the open subsets of X . Different classes of functions determine a different geometry on X .

This is very vague, so let's make this discussion more mathematical.

3.1. The model of differential geometry. The basic model of geometry that we have in mind is the one of differential geometry: here the space is the unit ball $X = B \subseteq \mathbf{R}^n$

$$B = \{x \in \mathbf{R}^n \mid \|x\| < 1\}$$

or more generally any smooth manifold. The functions are the smooth (C^∞) functions. More precisely, for each open set $U \subseteq X$ we have a set of smooth functions:

$$\mathcal{C}_X^\infty(U) = \{f: U \rightarrow \mathbf{R} \mid f \text{ is smooth} \}.$$

This is naturally a ring, since the sum and multiplication of two smooth functions is still smooth. Furthermore the constant functions are always smooth so this is a ring with unity. Notice that we can restrict smooth functions to open subsets: if $V \subseteq U$ are open subsets and f is smooth on U , the restriction $f|_V$ is smooth on V . This gives a ring homomorphism

$$\text{res}_V^U: \mathcal{C}_X^\infty(U) \rightarrow \mathcal{C}_X^\infty(V), \quad f \mapsto f|_V.$$

Notice that this satisfies the natural properties:

$$\text{res}_U^U = \text{id}, \quad \text{and} \quad \text{res}_W^U = \text{res}_W^V \circ \text{res}_V^U, \quad \text{if } W \subseteq V \subseteq U.$$

Furthermore, we have other important properties:

- (1) Functions are completely determined by what happens locally. If $U \subseteq X$ is an open subset with an open cover $U = \bigcup_i U_i$ and if $f, g \in \mathcal{C}_X^\infty(U)$ are such that $f|_{U_i} = g|_{U_i}$ for all i , then $f = g$.
- (2) Finally if we have a bunch of locally defined functions that are compatible, they can be glued together to a globally defined function: if $U \subseteq X$ is an open subset with an open cover $U = \bigcup_i U_i$ and if we have $f_i \in \mathcal{C}_X^\infty(U_i)$ such that $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ for all i, j , then there exists $f \in \mathcal{C}_X^\infty(U)$ such that $f|_{U_i} = f_i$ for all i .

This means that the C^∞ functions define a *sheaf of rings* on X . We recall briefly what sheaves are, but you should be familiar with the concept and its basic properties.

3.2. A short reminder about sheaves.

Definition 3.1 (Sheaf). Let X be a topological space. A *sheaf of sets/abelian groups/rings/...* $\mathcal{F}$ on X consists of

- a set/abelian group/ring/... $\mathcal{F}(U)$ for every open subset $U \subseteq X$
- a morphism of sets/groups/rings $\text{res}_V^U: \mathcal{F}(U) \rightarrow \mathcal{F}(V)$, written $f \mapsto f|_V$, for every pair of open subsets $V \subseteq U$. These are called restriction maps and we also ask that $\text{res}_U^U = \text{id}$, and $\text{res}_W^U = \text{res}_W^V \circ \text{res}_V^U$ whenever $W \subseteq V \subseteq U$.

Furthermore, we ask for two further conditions:

- (a) (Locality) If $U \subseteq X$ is open and $U = \bigcup U_i$ is an open cover, if $f, g \in \mathcal{F}(U)$ satisfy $f|_{U_i} = g|_{U_i}$ for all i , then $f = g$.
- (b) (Gluing) If $U \subseteq X$ is open and $U = \bigcup U_i$ is an open cover, if $f_i \in \mathcal{F}(U_i)$ satisfy $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ for all i, j , then there exists $f \in \mathcal{F}(U)$ such that $f|_{U_i} = f_i$ for all i .

The same data satisfying only the conditions on the restriction maps, without requiring (a) and (b), is called a *presheaf*.

Definition 3.2. If $\mathcal{F}$ is a sheaf on X and $U \subseteq X$ is open, the elements in $\mathcal{F}(U)$ are called *sections* of $\mathcal{F}$ on U . We will use the notations $\mathcal{F}(U) = \Gamma(U, \mathcal{F}) = H^0(U, \mathcal{F})$ to denote the set of sections.

Exercise 3.3. Assume that $\mathcal{F}$ is a sheaf of abelian groups or rings on a topological space X . Show that $\mathcal{F}(\emptyset) = 0$ is the zero abelian group or ring. $\diamond$

Example 3.4. By the properties above, $\mathcal{C}_X^\infty$ is a sheaf of rings on $X = B \subseteq \mathbf{R}^n$ the unit ball, and more generally on any manifold X .

If we look at sections of a sheaf on an arbitrarily small neighborhood of a point p we get germs and stalks:

Definition 3.5 (Germ and stalk). Let $\mathcal{F}$ be a sheaf (or presheaf) of sets/abelian groups/rings/... on X and let $p \in X$. Consider pairs (U, f) where $U \subseteq X$ is an open neighborhood of p and $f \in \mathcal{F}(U)$. Two pairs (U, f) and (V, g) are equivalent if there is an open neighborhood $W \subseteq U \cap V$ of p such that $f|_W = g|_W$. The equivalence class of a pair (U, f) is called the *germ* of f at p . The set of all germs at p is the *stalk* $\mathcal{F}_p$ of $\mathcal{F}$ at p .

Informally, we think of a germ in $\mathcal{F}_p$ as a section of $\mathcal{F}$ on an arbitrarily small open neighborhood of p .

Exercise 3.6. If $\mathcal{F}$ is a sheaf on X prove that a section on an open $U \subseteq X$ is determined by its stalks at all points of U . More precisely, prove that the map

$$\mathcal{F}(U) \rightarrow \prod_{p \in U} \mathcal{F}_p, \quad f \mapsto (\text{germ of } f \text{ at } p)_{p \in U}$$

is injective. You will need to use one of the defining properties of a sheaf. The other property can help you in describing the image. $\diamond$

We also notice that if $\mathcal{F}$ is a sheaf of sets/abelian groups/rings/... then the stalk $\mathcal{F}_p$ is also a set/abelian group/ring/...: for example, if $\mathcal{F}$ is a sheaf of rings, to add or multiply two germs, we represent them by (U, f) and (V, g) , restrict both to $U \cap V$, and add or multiply there. The result does not depend on the chosen representatives, because if two representatives agree on a smaller neighborhood of p then so do the sums and products, since the restriction maps are ring homomorphisms.

3.3. Locally ringed spaces. Let us look again at the sheaf $\mathcal{C}_X^\infty$ on $X = B \subseteq \mathbf{R}^n$ the unit ball in $\mathbf{R}^n$. The very good thing about functions is that they can be evaluated at points. For any $p \in X$, if $f \in \mathcal{C}_{X,p}^\infty$ is the germ of a smooth function around p , we can also evaluate it at p . This gives an evaluation map

$$\text{ev}_p: \mathcal{C}_{X,p}^\infty \rightarrow \mathbf{R} \quad f \mapsto f(p)$$

which is a ring homomorphism, and is surjective since it sends the germ of the constant function c to c . The kernel is a maximal ideal, because the quotient $\mathbf{R}$ is a field:

$$\mathfrak{m}_p = \{f \in \mathcal{C}_{X,p}^\infty \mid f(p) = 0\}.$$

Notice that this is the unique maximal ideal of $\mathcal{C}_{X,p}^\infty$, equivalently all germs not in $\mathfrak{m}_p$ are invertible. Indeed, if f is a smooth function defined in an open neighborhood U of p and if $f(p) \neq 0$, then up to shrinking the neighborhood we can assume that f never vanishes on U . Hence $1/f$ is a well-defined smooth function on U , so the germ of f is a unit of $\mathcal{C}_{X,p}^\infty$. This means that $\mathcal{C}_{X,p}^\infty$ is a *local ring* with maximal ideal $\mathfrak{m}_p$ and *residue field* $\mathcal{C}_{X,p}^\infty/\mathfrak{m}_p \cong \mathbf{R}$.

After this discussion, we can give a precise definition of geometric space, which is actually called a *locally ringed space*:

Definition 3.7 (Locally ringed space). A *locally ringed space* is a topological space X together with a sheaf of rings $\mathcal{O}_X$ such that for all $p \in X$ the stalk $\mathcal{O}_{X,p}$ is a local ring.

This definition is very general: in particular, we point out an important difference with the setting that we are used to:

Warning: If $U \subseteq X$ is open, the sections of $\mathcal{O}_X(U)$ are **not always functions** on U in the sense of maps from U to a set. This will happen in particular for schemes.

There is however a way to evaluate the elements of $\mathcal{O}_X(U)$ at $p \in U$ using the condition that $\mathcal{O}_{X,p}$ is a local ring. If $\mathfrak{m}_p$ is the maximal ideal of $\mathcal{O}_{X,p}$, the *residue field* at p is $\kappa(p) = \mathcal{O}_{X,p}/\mathfrak{m}_p$, and the projection is a homomorphism that we call the evaluation map:

$$\text{ev}_p: \mathcal{O}_{X,p} \rightarrow \mathcal{O}_{X,p}/\mathfrak{m}_p = \kappa(p), \quad f \mapsto f(p)$$

Even if we have this notion of evaluation, we cannot always think of the elements in $\mathcal{O}_X(U)$ as functions on U . Indeed, we first notice that the evaluation map at different points takes values in a priori different fields. Even more importantly:

Warning: A section $f \in \mathcal{O}_X(U)$ is **not always determined by its values** $f(p)$ at points $p \in U$. This will happen in particular for schemes.

This being said, let's give some examples where everything works as usual, meaning that sections are actually functions to a field.

Example 3.8. The following are locally ringed spaces:

- (1) A smooth manifold X with the Euclidean topology (e.g. $X = B \subseteq \mathbf{R}^n$ the open unit ball) and the sheaf of $\mathcal{C}^\infty$ functions.

$$\mathcal{C}_X^\infty(U) = \{f: U \rightarrow \mathbf{R} \mid f \text{ smooth function} \}.$$

- (2) A complex manifold X with the Euclidean topology (e.g. $X = B \subseteq \mathbf{C}^n$ the open unit ball) and the sheaf of complex analytic functions.

$$\mathcal{O}_X^{\text{an}}(U) = \{f: U \rightarrow \mathbf{C} \mid f \text{ analytic} \}.$$

The following is also an example, but is presented as an exercise in case you do not know it already.

Exercise 3.9. Let k be an algebraically closed field and $X \subseteq k^n$ an affine algebraic set. Show that X with the sheaf of regular functions

$$\mathcal{O}_X(U) = \{f: U \rightarrow k \mid f \text{ regular function} \}.$$

is a locally ringed space. More precisely, recall that a point $p = (a_1, \dots, a_n) \in X$ corresponds to the maximal ideal $\mathfrak{m}_p = (x_1 - a_1, \dots, x_n - a_n) \subseteq \mathcal{O}(X)$ and prove that the stalk $\mathcal{O}_{X,p}$ is isomorphic to the localization of the ring $\mathcal{O}(X)$ at the maximal ideal: $\mathcal{O}_{X,p} \cong \mathcal{O}(X)_{\mathfrak{m}_p}$. $\diamond$

Example 3.10. If $(X, \mathcal{O}_X)$ is a locally ringed space and $U \subseteq X$ is an open subset, then this is also a locally ringed space with the sheaf $\mathcal{O}_U$ defined by $\mathcal{O}_U(U') = \mathcal{O}_X(U')$ for any open subset $U' \subseteq U$ and the corresponding restriction maps.

3.4. Morphisms of locally ringed spaces. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be two locally ringed spaces. What should a morphism of locally ringed spaces from X to Y be? Well, we know that the structure of locally ringed space is given by a topological space X with sections $\mathcal{O}_X(U)$. So a morphism $F: X \rightarrow Y$ should describe what points do and also what sections do.

Let's go back again to differential geometry and let's look at $X = B \subseteq \mathbf{R}^n$ the open unit ball (or any manifold X). Recall that a smooth map $F: X \rightarrow \mathbf{R}^m$ is a map given in coordinates by $F(p) = (F_1(p), \dots, F_m(p))$, such that all the $F_i: X \rightarrow \mathbf{R}$ are smooth functions. We want to reinterpret this from the point of view of locally ringed spaces.

Exercise 3.11. Show that a map $F: X \rightarrow \mathbf{R}^m$ is smooth if and only if the two following conditions hold:

- (a) $F: X \rightarrow \mathbf{R}^m$ is a continuous map.
- (b) F pulls back smooth functions from $\mathbf{R}^m$ to smooth functions on X : this means that for any open subset $V \subseteq \mathbf{R}^m$ and for any smooth function $f: V \rightarrow \mathbf{R}$ its pull-back $f \circ F: F^{-1}(V) \rightarrow \mathbf{R}$ is a smooth function on $F^{-1}(V)$.

◇

We rephrase this in sheaf-theoretic terms.

Definition 3.12 (Morphism of sheaves). If X is a topological space with two sheaves $\mathcal{F}, \mathcal{G}$ of sets/abelian groups/rings/... a morphism $\varphi: \mathcal{F} \rightarrow \mathcal{G}$ is a collection of morphisms of sets/abelian groups/rings/...

$$\varphi_U: \mathcal{F}(U) \rightarrow \mathcal{G}(U), \quad U \subseteq X \text{ open},$$

that are compatible with the restriction maps: for any $V \subseteq U$ open subset the following diagram commutes:

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\varphi_U} & \mathcal{G}(U) \\ \text{res}_V^U \downarrow & & \downarrow \text{res}_V^U \\ \mathcal{F}(V) & \xrightarrow{\varphi_V} & \mathcal{G}(V) \end{array}$$

Definition 3.13 (Pushforward of a sheaf). Let $F: X \rightarrow Y$ be a continuous map of topological spaces and let $\mathcal{F}$ be a sheaf of sets/rings/abelian groups/... on X . The *pushforward* of $\mathcal{F}$ along F is the sheaf of sets/abelian groups/rings/... on Y defined by

$$F_*\mathcal{F}(V) = \mathcal{F}(F^{-1}(V)), \quad \text{for any } V \subseteq Y \text{ open subset.}$$

and the restriction maps are those of $\mathcal{F}$.

Exercise 3.14. Show that $F_*\mathcal{F}$ is indeed a sheaf of sets/abelian groups/rings/... if $\mathcal{F}$ is. ◇

With this notation we can rephrase exercise 3.11 by saying that if $X = \mathbf{R}^n$ (or X any smooth manifold), a smooth map $F: X \rightarrow \mathbf{R}^m$ is the same as a continuous map $F: X \rightarrow \mathbf{R}^m$ such that the pullback defines a morphism of sheaves on $\mathbf{R}^m$

$$F^\#: \mathcal{C}_{\mathbf{R}^m}^\infty \rightarrow F_* \mathcal{C}_X^\infty, \quad \mathcal{C}_{\mathbf{R}^m}^\infty(V) \rightarrow \mathcal{C}_X^\infty(F^{-1}(V)), \quad f \mapsto f \circ F$$

Let $p \in X$ be a point that maps to $q = F(p) \in \mathbf{R}^m$. Then the germ of a smooth function around q pulls back to the germ of a smooth function around p . Furthermore, if the germ vanishes at q then its pullback vanishes at p . This means that we have a ring homomorphism

$$F^\#: \mathcal{C}_{\mathbf{R}^m, q}^\infty \rightarrow \mathcal{C}_{X, p}^\infty, \text{ such that } F^\#(\mathfrak{m}_q) \subseteq \mathfrak{m}_p.$$

This discussion leads to the definition of a morphism between locally ringed spaces:

Definition 3.15 (Morphism of locally ringed spaces). A morphism from a locally ringed space $(X, \mathcal{O}_X)$ to a locally ringed space $(Y, \mathcal{O}_Y)$ is given by a pair $(F, F^\#)$ where:

- $F: X \rightarrow Y$ is a continuous map of topological spaces.
- $F^\#: \mathcal{O}_Y \rightarrow F_* \mathcal{O}_X$ is a morphism of sheaves of rings (sometimes called the pullback morphism), such that if we take any $p \in X$, $q = F(p) \in Y$ the induced map $F^\#: \mathcal{O}_{Y, q} \rightarrow \mathcal{O}_{X, p}$ is a homomorphism of local rings, meaning that $F^\#(\mathfrak{m}_q) \subseteq \mathfrak{m}_p$.

We will often denote such a morphism only via $F: X \rightarrow Y$, but we have to keep in mind the pullback morphism $F^\#: \mathcal{O}_Y \rightarrow F_* \mathcal{O}_X$ as well.

Notice that if $F: X \rightarrow Y$ is a morphism of locally ringed spaces and if $p \in X$ maps to $q = F(p) \in Y$, then we have an induced extension of fields:

$$F^\#: \kappa(q) = \mathcal{O}_{Y, q} / \mathfrak{m}_q \hookrightarrow \mathcal{O}_{X, p} / \mathfrak{m}_p = \kappa(p)$$

that makes the following diagram commute:

$$\begin{array}{ccc} \mathcal{O}_{Y, q} & \xrightarrow{F^\#} & \mathcal{O}_{X, p} \\ \text{ev}_q \downarrow & & \downarrow \text{ev}_p \\ \kappa(q) & \xrightarrow{F^\#} & \kappa(p) \end{array}$$

We interpret the diagram as saying that if f is a section of $\mathcal{O}_Y$ around q , the value of the pullback $F^\#(f)$ at p is exactly the value of f at q , as with the usual pull-back of functions.

Remark 3.16. It is important to notice that a morphism of locally ringed spaces consists of information on where the points go and on where the sections $\mathcal{O}_Y(U)$ go. It is not enough to specify the continuous map $F: X \rightarrow Y$: indeed, the elements in $\mathcal{O}_Y(U)$ are not functions in general, so that they are not determined by their values at points.

Example 3.17. If $(X, \mathcal{O}_X)$ is a locally ringed space, the identity $\text{id}_X: X \rightarrow X$ as a morphism of locally ringed spaces is given by

- The identity map at the level of sets $\text{id}_X: X \rightarrow X$.
- The identity map at the level of sheaves $\text{id}_X: \mathcal{O}_X \rightarrow \text{id}_{X,*} \mathcal{O}_X = \mathcal{O}_X$.

Finally, we note that if $(X, \mathcal{O}_X), (Y, \mathcal{O}_Y), (Z, \mathcal{O}_Z)$ are three locally ringed spaces and if $F: X \rightarrow Y$ and $G: Y \rightarrow Z$ are morphisms of locally ringed spaces, then we define the composition $(G \circ F): X \rightarrow Z$ as the morphism of locally ringed spaces given by

- The composition of the maps of sets $G \circ F: X \rightarrow Z$.
- The pullback morphism $(G \circ F)^\#: \mathcal{O}_Z \rightarrow (G \circ F)_* \mathcal{O}_X$ defined on each open subset $W \subseteq Z$ as the composition of the pullbacks

$$\mathcal{O}_Z(W) \xrightarrow{G^\#} \mathcal{O}_Y(G^{-1}(W)) \xrightarrow{F^\#} \mathcal{O}_X(F^{-1}(G^{-1}(W)))$$

With this composition and the identity, locally ringed spaces form a category. In particular there is a notion of *isomorphism* of locally ringed spaces: this is as usual a morphism $F: X \rightarrow Y$ such that there is another morphism $G: Y \rightarrow X$ such that

$$F \circ G = \text{id}_Y, \quad G \circ F = \text{id}_X.$$

If you know what smooth or complex manifolds are with the usual definition via charts, the following is a very good exercise:

Exercise 3.18. Let $(X, \mathcal{O}_X)$ be a locally ringed space with an open cover $X = \bigcup U_i$ such that each locally ringed space $(U_i, \mathcal{O}_{U_i})$ is isomorphic to the unit open ball $B \subseteq \mathbf{R}^n$ with the sheaf of smooth functions $(U_i, \mathcal{O}_{U_i}) \cong (B, \mathcal{C}_B^\infty)$. Show that X has the structure of a smooth manifold such that $\mathcal{O}_X$ can be seen as the sheaf of smooth functions $\mathcal{C}_X^\infty$. If instead each $(U_i, \mathcal{O}_{U_i})$ is isomorphic to a unit ball $B \subseteq \mathbf{C}^n$ with the sheaf of complex analytic functions $\mathcal{O}_B^{\text{an}}$, show that X has the structure of a complex manifold. If in your definition of smooth or complex manifolds they must be Hausdorff and second countable then you should add these hypotheses to X as well. $\diamond$

As with smooth or complex manifolds, we will see that a scheme is a locally ringed space $(X, \mathcal{O}_X)$ which is locally isomorphic to a particular locally ringed space, called an affine scheme.

4. AFFINE SCHEMES

Affine schemes are the basic building blocks of algebraic geometry. We want to define them as locally ringed spaces, meaning that we need a topological space and a sheaf of functions. We want to do it in a way that there is a perfect correspondence between algebra and geometry.

4.1. Affine algebraic sets and the maximal spectrum. To motivate the construction let us go back again to the case of an affine algebraic set $X \subseteq k^n$ over an algebraically closed field k . As a consequence of the Nullstellensatz as in Theorem 2.6 we see that:

- The points of X correspond to the maximal ideals of $\mathcal{O}(X)$. If $p \in X$ is a point, the corresponding maximal ideal $\mathfrak{m}_p \subseteq \mathcal{O}(X)$ is the kernel of the evaluation map

$$\text{ev}_p: \mathcal{O}(X) \longrightarrow k, \quad f \mapsto f(p)$$

This means that $f(p) = 0$ if and only if $f \in \mathfrak{m}_p$.

Hence, we can identify:

$$X = \text{SpecMax } \mathcal{O}(X) = \{ \text{maximal ideals in } \mathcal{O}(X) \}$$

- The closed subsets for the Zariski topology on X are defined from an ideal $I \subseteq \mathcal{O}(X)$ by taking

$$\begin{aligned} V(I) &= \{p \in X \mid f(p) = 0 \text{ for all } f \in I\} \\ &= \{\mathfrak{m} \in \text{SpecMax } \mathcal{O}(X) \mid f \in \mathfrak{m} \text{ for all } f \in I\} \\ &= \{\mathfrak{m} \in \text{SpecMax } \mathcal{O}(X) \mid I \subseteq \mathfrak{m}\} \end{aligned}$$

On the level of open subsets, this means that every open subset is an union of open subsets of the form

$$D(f) = \{p \in X \mid f(p) \neq 0\} = \{\mathfrak{m} \in \text{SpecMax } \mathcal{O}(X) \mid f \notin \mathfrak{m}\}$$

- Finally the sheaf of regular functions on X is characterized by the fact that the elements in $\mathcal{O}(X)$ are the regular functions on the whole of X and that the elements in the localization

$$\mathcal{O}(X)_f = \left\{ \frac{g}{f^n} \mid g \in \mathcal{O}(X), n \in \mathbf{N} \right\}$$

are regular functions on $D(f)$.

This way, we have recast the space X purely in terms of the ring $\mathcal{O}(X)$. Affine schemes do this for every ring.

4.2. The spectrum of a ring as a topological space. One key insight of schemes is that one should not look only at maximal ideals, but rather all prime ideals. We will justify this later:

Definition 4.1 (Spectrum of a ring). Let R be a ring. The *spectrum* of R is the set

$$\text{Spec } R = \{\mathfrak{p} \subseteq R \mid \mathfrak{p} \text{ is a prime ideal}\}.$$

To be clear, we will sometime use the notation $[\mathfrak{p}]$ when we look at the prime ideal $\mathfrak{p} \subseteq R$ as a point in $\text{Spec } R$.

Remark 4.2. Recall that the trivial ideal $(1) = R$ is never prime by definition.

Example 4.3 (The empty set). The zero ring has no prime ideals, hence $\text{Spec } 0 = \emptyset$. Conversely, if R is a non-zero ring, then Zorn's lemma shows that it has always a maximal ideal, which in particular is prime, hence $\text{Spec } R = \emptyset$ if and only if $R = 0$.

Example 4.4 ($\text{Spec } k$). A field k has only two ideals, (0) and k : indeed any non-zero ideal contains a unit, hence contains 1. The ideal (0) is prime, so $\text{Spec } k = \{[(0)]\}$ is a single point.

Example 4.5 ($\text{Spec } \mathbf{C}[x]$). The ring $\mathbf{C}[x]$ is a principal ideal domain hence the non-zero prime ideals are those generated by irreducible polynomials. By the fundamental theorem of algebra, the irreducible polynomials in $\mathbf{C}[x]$ are exactly the polynomials of degree 1, and $(x - a) = (x - b)$ if and only if $a = b$. Furthermore, the zero ideal is prime so that

$$\text{Spec } \mathbf{C}[x] = \{[(0)]\} \cup \{[(x - a)] \mid a \in \mathbf{C}\}.$$

We can think of this as the usual affine line $\mathbf{C}$ together with an extra point corresponding to the ideal (0) .

Example 4.6 ($\text{Spec } \mathbf{R}[x]$). The ring $\mathbf{R}[x]$ is a principal ideal domain, and a non-zero polynomial is irreducible over $\mathbf{R}[x]$ if and only if it has degree one or if is of degree two and its complex roots are two non-real complex conjugate numbers $(x - z)(x - \bar{z})$ for $z \in \mathbf{C} \setminus \mathbf{R}$. Hence

$$\text{Spec } \mathbf{R}[x] = \{(0)\} \cup \{(x - a) \mid a \in \mathbf{R}\} \cup \{((x - z)(z - \bar{z})) \mid z \in \mathbf{C} \setminus \mathbf{R}\}.$$

Notice that the points distinct from $\{0\}$ correspond to the classes of complex numbers $\mathbf{C}$ modulo the complex conjugation.

Example 4.7 ($\text{Spec } \mathbf{Z}$). Any non-zero prime ideal in $\mathbf{Z}$ is generated by a prime number

$$\text{Spec } \mathbf{Z} = \{(0)\} \cup \{(p) \mid p \text{ a prime number}\}.$$

Let R be a ring and $p = [\mathfrak{p}] \in \text{Spec } R$ a point corresponding to a prime ideal $\mathfrak{p} \subseteq R$. The *residue field* at p is

$$(4.1) \quad \kappa(p) = \text{Frac } R/\mathfrak{p} = R_{\mathfrak{p}}/\mathfrak{p}_{\mathfrak{p}}$$

and we also call the projection $R_{\mathfrak{p}} \rightarrow \kappa(p)$ the *evaluation map* at p and we write it as

$$\text{ev}_p: R_{\mathfrak{p}} \longrightarrow \kappa(p), \quad f \mapsto f(p) \in \kappa(p)$$

In particular, we can define the *value of an element* $f \in R$ at $[p]$ to be the image of f via the composition

$$R \rightarrow R_{\mathfrak{p}} \rightarrow \kappa(p), \quad f \mapsto f(p)$$

In particular we see that if $f \in R$ then

$$f(p) = 0 \iff f \in \mathfrak{p}$$

Remark 4.8. In general the element $f \in R$ is not uniquely determined by its value at all points $[p] \in \text{Spec } R$. For example, consider $\text{Spec } \mathbf{C}[x]/(x^2) = \{[(x)]\}$ and the element $x \in \mathbf{C}[x]/(x^2)$. This is non-zero in $\mathbf{C}[x]/(x^2)$, but vanishes at the unique point of $\text{Spec } \mathbf{C}[x]/(x^2)$. We are going to see later that the issue is that $x \in \mathbf{C}[x]/(x^2)$ is non-zero but nilpotent.

Via the evaluation of elements in R , we can define the Zariski topology on $\text{Spec } R$: for an ideal $I \subseteq R$ define the corresponding closed subset as

$$V(I) = \{p \in \text{Spec } R \mid f(p) = 0 \text{ for all } f \in I\} = \{\mathfrak{p} \in \text{Spec } R \mid I \subseteq \mathfrak{p}\}.$$

if $f_1, \dots, f_n \in R$ we will also write $V(f_1, \dots, f_n) = V((f_1, \dots, f_n))$.

Lemma 4.9. *Let R be a ring. Then the following properties hold:*

- (1) $V(0) = \text{Spec } R$ and $V(R) = \emptyset$.
- (2) If $I, J \subseteq R$ are ideals, then $V(I) \cup V(J) = V(I \cap J) = V(IJ)$.
- (3) If $\{I_{\lambda}\}_{\lambda \in \Lambda}$ is a family of ideals then $\bigcap_{\lambda} V(I_{\lambda}) = V(\sum_{\lambda} I_{\lambda})$.

In particular the sets $V(I)$ are the closed subsets of a topology on $\text{Spec } R$.

Proof. (1) Every ideal contains 0, and no prime ideal contains 1.

(2) Since $IJ \subseteq I \cap J \subseteq I$ and $I \cap J \subseteq J$, we have $V(I) \cup V(J) \subseteq V(I \cap J) \subseteq V(IJ)$. Conversely let $\mathfrak{p} \in V(IJ)$ and suppose that $\mathfrak{p} \notin V(I)$, so there is $a \in I \setminus \mathfrak{p}$. For every $b \in J$ we have $ab \in IJ \subseteq \mathfrak{p}$, hence $b \in \mathfrak{p}$ since $\mathfrak{p}$ is prime and $a \notin \mathfrak{p}$. Thus $J \subseteq \mathfrak{p}$ and $\mathfrak{p} \in V(J)$.

(3) A prime ideal $\mathfrak{p}$ contains all the I_λ if and only if it contains the ideal $\sum_\lambda I_\lambda$ that they generate. $\square$

Definition 4.10 (Zariski topology on $\text{Spec } R$). The Zariski topology on $\text{Spec } R$ has the sets $V(I)$ for $I \subseteq R$ an ideal as closed subsets

By construction every closed subset in $\text{Spec}(R)$ is an intersection of closed subsets of the form $V(f)$ for $f \in R$. Hence the complements $\text{Spec } R \setminus V(f)$ form a basis of open sets for the Zariski topology. These are particularly nice open sets and we give them a name:

Definition 4.11 (Principal open subsets). For $f \in R$ the *principal open subset* (or also distinguished open subset) of f is

$$D(f) = \text{Spec } R \setminus V(f) = \{p \in \text{Spec } R \mid f(p) \neq 0\} = \{\mathfrak{p} \in \text{Spec } R \mid f \notin \mathfrak{p}\}.$$

The principal open subsets are a basis of the Zariski topology of $\text{Spec } R$.

Example 4.12 ($\text{Spec } k$). If k is a field, then $\text{Spec } k$ is a single point which is closed.

Example 4.13 ($\text{Spec } \mathbf{C}[x]$). We have seen that

$$\text{Spec } \mathbf{C}[x] = \{(0)\} \cup \{(x - a) \mid a \in \mathbf{C}\}.$$

For the points $[(x - a)]$ with $a \in \mathbf{C}$ we see that $V(x - a) = \{[(x - a)]\}$, indeed since the ideal $(x - a) \subseteq \mathbf{C}[x]$ is maximal, any prime ideal that contains it must be $(x - a)$ itself. In particular the points $[(x - a)]$ are closed. The point $[0]$ instead is not closed: if $[0] \in V(I)$ for an ideal I , then $I \subseteq (0)$ so that $I = 0$ and then $V(I) = \text{Spec } R$. This shows that the closure of $[0]$ is the whole space $\text{Spec } R$ (!).

In general, closed points correspond to maximal ideals. We can be more precise and compute explicitly the closure of a point:

Lemma 4.14. *Let R be a ring and $[\mathfrak{p}] \in \text{Spec } R$ a point. Show that the closure of this point is given by*

$$\overline{\{[\mathfrak{p}]\}} = V(\mathfrak{p})$$

In particular, show that $[\mathfrak{p}]$ is a closed point if and only if $\mathfrak{p}$ is a maximal ideal.

Proof. By definition the closure of $[\mathfrak{p}]$ is the intersection of all closed subsets containing $[\mathfrak{p}]$, and if $I \subseteq R$ is an ideal, we see that $[\mathfrak{p}] \in V(I)$ if and only if $I \subseteq \mathfrak{p}$. Hence

$$\overline{\{[\mathfrak{p}]\}} = \bigcap_{[\mathfrak{p}] \in V(I)} V(I) = \bigcap_{I \subseteq \mathfrak{p}} V(I) = V\left(\sum_{I \subseteq \mathfrak{p}} I\right) = V(\mathfrak{p}).$$

Recall now that any proper ideal in a ring is always contained in a maximal ideal, in particular a prime ideal $\mathfrak{p}$ is maximal if and only if $V(\mathfrak{p}) = \{[\mathfrak{p}]\}$, meaning that $[\mathfrak{p}]$ is a closed point. $\square$

Example 4.15 ($\text{Spec } \mathbf{Z}$). We have that

$$\text{Spec } \mathbf{Z} = \{(0)\} \cup \{(p) \mid p \text{ a prime number}\}.$$

All the points $[(p)]$ for p a prime number are closed, while the closure of point $[(0)]$ is the whole $\text{Spec } \mathbf{Z}$.

Exercise 4.16. If R is a principal ideal domain, show that any closed subset in $\text{Spec } R$ is either the whole space or a finite union of closed points. $\diamond$

Example 4.17 ($\text{Spec } \mathbf{C}[x, y]$). This is a more interesting example. Every prime ideal in $\mathbf{C}[x, y]$ is one of the following three types:

- $(x - a, y - b)$ for $a, b \in \mathbf{C}$. These are all the maximal ideals because of the Nullstellensatz.
- $(f(x, y))$ for $f(x, y) \in \mathbf{C}[x, y]$ an irreducible polynomial.
- The zero ideal (0) .

We then see that if $f = f(x, y)$ is an irreducible polynomial, then

$$V(f) = \{[(x - a, y - b)] \mid (a, b) \in \mathbf{C}^2, f(a, b) = 0\} \cup \{[(f)]\}$$

and that $[(f)]$ is dense in $V(f)$. Furthermore $[(0)]$ is dense on the whole space $\text{Spec } \mathbf{C}[x, y]$. Then we can think of $\text{Spec } \mathbf{C}[x, y]$ as the usual complex plane $\mathbf{C}^2$ together with one special fuzzy point on each irreducible curve, which is dense in this curve, and another more special fuzzy point which is dense on the whole space. A closed subset of $\mathbf{C}[x, y]$ is either the whole space or a finite union of irreducible curves (together with their special fuzzy points) and closed points.

4.3. The topological Nullstellensatz. For $\text{Spec } R$ there is an easy version of the Nullstellensatz, which is equivalent to this basic fact in commutative algebra:

Lemma 4.18. *If $I \subseteq R$ is an ideal and if $\sqrt{I}$ is its radical, then*

$$\sqrt{I} = \bigcap_{\substack{\mathfrak{p} \subseteq R \text{ prime} \\ \mathfrak{p} \supseteq I}} \mathfrak{p} = \bigcap_{[\mathfrak{p}] \in V(I)} \mathfrak{p}$$

Proof. Up to replacing R with R/I , we can assume that $I = (0)$. We have to prove that $\sqrt{(0)}$ is the intersection of all prime ideals in R . We first note that $f \in \sqrt{(0)}$ if and only if $R_f = 0$. Indeed $R_f = 0$ if and only if $1 = 0$ in R_f , meaning that there is $n \in \mathbf{N}$ such that $f^n = f^n \cdot 1 = f \cdot 0 = 0$. Since a ring is non-zero if and only if it has a prime ideal, we see that $f \notin \sqrt{(0)}$ if and only if there is a prime ideal in R_f , which corresponds to a prime ideal $\mathfrak{p} \subseteq R$ such that $f \notin \mathfrak{p}$. $\square$

We can interpret it as follows: for a subset $S \subseteq \text{Spec } R$ we define its vanishing ideal as

$$I(S) = \bigcap_{\mathfrak{p} \in S} \mathfrak{p}, \quad I(\emptyset) = R$$

This is always a radical ideal, since it is an intersection of prime (and hence radical) ideals. This is the analog of the vanishing ideal of an algebraic set in k^n . The analog, but much easier, version of the Nullstellensatz is

Proposition 4.19 (Topological Nullstellensatz). *Let R be a ring.*

- (1) *If $I \subseteq R$ is an ideal $V(I) = V(\sqrt{I})$. and $I(V(I)) = \sqrt{I}$.*
- (2) *The following maps are inclusion-reversing bijections, and inverse to each other.*

$$\begin{array}{ccc} \{\text{closed subsets of } \text{Spec } R\} & \longleftrightarrow & \{\text{radical ideals } I \subseteq R\} \\ Z & \longmapsto & I(Z) \\ V(I) & \longleftarrow & I \end{array}$$

Proof. Point (1) is just a reformulation of Lemma 4.18. For point (2) it is clear that the maps are well-defined and inclusion-reversing. Thanks to point (1), we are left to show that if $ZV \subseteq \operatorname{Spec} R$ is a closed subset, then $V(I(Z)) = Z$. Write $Z = V(J)$. If $\mathfrak{p} \in Z$ then $I(Z) \subseteq \mathfrak{p}$ by definition of $I(Z)$, so $\mathfrak{p} \in V(I(Z))$. Conversely, every $\mathfrak{p} \in Z = V(J)$ contains J , so $J \subseteq I(Z)$, hence $V(I(Z)) \subseteq V(J) = Z$. $\square$

Under this bijection, the empty set corresponds to the ideal R , and $\operatorname{Spec} R$ corresponds to the nilradical of R :

Definition 4.20 (Nilradical). Let R be a ring. An element $f \in R$ is called *nilpotent* if $f^n = 0$ for a certain $n \in \mathbf{N}$. The *nilradical* of R is:

$$N(R) = \sqrt{0} = \{f \in R \mid f \text{ nilpotent}\} = \bigcap_{\mathfrak{p} \in \operatorname{Spec} R} \mathfrak{p}.$$

The ring R is called reduced if and only if $N(R) = (0)$.

Exercise 4.21. Let R be a ring and $f \in R$.

- (i) Show that $D(f) = \emptyset$ and that $V(f) = \operatorname{Spec} R$ if and only if f is nilpotent.
- (ii) Show that if $R \neq 0$ then $D(f) = \operatorname{Spec} R$ if and only if f is a unit.

◇

4.4. The sheaf of regular functions. Now we want to construct a sheaf of rings $\mathcal{O}_{\operatorname{Spec} R}$ on $\operatorname{Spec} R$ that gives it the structure of a locally ringed space. Following the model of algebraic sets over an algebraically closed field, the main property that we want is that on any principal open set $D(f) \subseteq \operatorname{Spec} R$:

$$\mathcal{O}_{\operatorname{Spec} R}(D(f)) = R_f = \left\{ \frac{a}{f^n} \mid a \in R, n \in \mathbf{N} \right\}$$

We discuss a couple properties of principal open subsets:

Lemma 4.22. *If R is a ring and $f, g \in R$, then in $\operatorname{Spec} R$ it holds that*

- (1) $D(f) = D(f^n)$ for any $n \in \mathbf{N}$
- (2) $D(f) \cap D(g) = D(fg)$.
- (3) $D(f) \subseteq D(g)$ if and only if $f \in \sqrt{(g)}$.

Proof. (1) This is equivalent to $V(f) = V(f^n)$, which is true by Proposition 4.19
 (2) $V(f \cdot g) = V(f) \cup V(g)$. Otherwise observe that for any point $p \in \operatorname{Spec} R$ then $(fg)(p) = f(p) \cdot g(p) \neq 0$ in $\kappa(p)$ if and only if $f(p) \neq 0$ and $g(p) \neq 0$ because $\kappa(p)$ is a field.
 (3) We have that $D(f) \subseteq D(g)$ if and only if $V(f) \supseteq V(g)$ which is equivalent to $\sqrt{f} \subseteq \sqrt{g}$ because of Proposition 4.19. $\square$

Consider then principal open subsets $D(f) \subseteq D(g)$: Lemma 4.22 shows that there is an $n \in \mathbf{N}$ such that $f^n = h \cdot g$ for a certain $h \in R$. In particular if we consider the natural map $R \rightarrow R_f, a \mapsto \frac{a}{1}$, we see that $\frac{g}{1}$ is invertible in R_f because

$$\frac{g}{1} \cdot \frac{h}{f^n} = \frac{gh}{f^n} = 1 \quad \text{hence we write } \frac{1}{g} = \left(\frac{g}{1}\right)^{-1} = \frac{h}{f^n}$$

Then we have a natural homomorphism of rings

$$(4.2) \quad R_g \longrightarrow R_f, \quad \frac{a}{g^m} \mapsto \frac{a}{g^m} = \frac{a}{1} \cdot \left(\frac{1}{g}\right)^m$$

These are going to be the restriction maps of a sheaf of rings on $\text{Spec } R$. Before constructing the sheaf, we need another useful result

Lemma 4.23. *Let $(g_i)_{i \in I}$ be a family of elements $g_i \in R$. Then it holds that*

- (1) *If $f \in R$, then $D(f) \subseteq \bigcup D(g_i)$ if and only if the $\frac{g_i}{1} \in R_f$ generate the whole ring R_f as an ideal. This is equivalent to the existence of $n \in \mathbf{N}$ such that*

$$f^n = h_{i_1}g_{i_1} + \dots + h_{i_r}g_{i_r} \quad \text{for certain } h_{i_1}, \dots, h_{i_r} \in R$$

- (2) *$\text{Spec } R = \bigcup D(g_i)$ if and only if the g_i generate the whole ring R as an ideal. This is equivalent to*

$$1 = h_{i_1}g_{i_1} + \dots + h_{i_r}g_{i_r} \quad \text{for certain } h_{i_1}, \dots, h_{i_r} \in R$$

Proof. For point (1) we observe that $D(f) \subseteq \bigcup D(g_i)$ if and only if $V(f) \supseteq V(g_i)$ which is equivalent to $f \in \sqrt{(g_i)}$. Equivalently, there is $n \in \mathbf{N}$ such that

$$f^n = h_{i_1}g_{i_1} + \dots + h_{i_r}g_{i_r} \quad \text{for certain } h_{i_1}, \dots, h_{i_r} \in R$$

If this holds, then

$$1 = \frac{h_{i_1}}{f^n} \cdot \frac{g_{i_1}}{1} + \dots + \frac{h_{i_r}}{f^n} \cdot \frac{g_{i_r}}{r} \quad \text{in } R_f$$

so that the $\frac{g_i}{1}$ generate the whole ring R_f . A similar reasoning, but in reverse shows the converse implication. Point (2) is shown by taking $f = 1$ in point (1). $\square$

As a corollary, we see that all principal open subsets in $\text{Spec } R$ are compact. For historical reasons, See <https://mathoverflow.net/questions/16971/compact-and-quasi-compact> for a discussion. In algebraic geometry this is actually called quasi-compactness:

Definition 4.24. A topological space X is called quasi-compact if every open cover of X has a finite subcover.

Corollary 4.25. *Any principal open subset $D(f) \subseteq \text{Spec } R$, $f \in R$ is quasi-compact. In particular $\text{Spec } R$ is quasi-compact.*

Proof. Since the principal open subsets are a basis of the topology, it is enough to show that if $D(f)$ is covered by principal open subsets $D(g_i)$, it is then covered by finitely many of them. If $D(f)$ is covered by principal open subsets $D(g_i)$, Lemma 4.23 shows that

$$f^n = h_{i_1}g_{i_1} + \dots + h_{i_r}g_{i_r}$$

for a certain $n \in \mathbf{N}$ and certain $h_{i_1}, \dots, h_{i_r} \in R$. Then Lemma 4.23 again shows that $D(f) = D(g_{i_1}) \cup \dots \cup D(g_{i_r})$. $\square$

Remark 4.26. It is not always true that any open subset $U \subseteq \text{Spec } R$ is compact. We will see a counterexample later on.

Now we can construct the sheaf of rings that we want on $\text{Spec } R$.

Theorem 4.27. *There is a sheaf of rings $\mathcal{O}_{\text{Spec } R}$ on $\text{Spec } R$ such that*

$$\mathcal{O}_{\text{Spec } R}(D(f)) \cong R_f = \left\{ \frac{a}{f^n} \mid a \in R, n \in \mathbf{N} \right\} \text{ for any } f \in R$$

and whose restriction maps on principal open sets are given as in (4.2). Furthermore, for this sheaf it holds that $\mathcal{O}_{\text{Spec } R}(\text{Spec } R) = R$ and there are natural isomorphisms of rings

$$R_{\mathfrak{p}} \longrightarrow \mathcal{O}_{\text{Spec } R, [\mathfrak{p}]}, \quad \text{for any } [\mathfrak{p}] \in \text{Spec } R.$$

In particular $(\text{Spec } R, \mathcal{O}_{\text{Spec } R})$ is a locally ringed space.

Proof. At a first reading is better to remember only the statement of this result rather than the proof, as it might be a bit confusing.

Let us write in the proof only $\mathcal{O}$ instead of $\mathcal{O}_{\text{Spec } R}$ for simplicity. We have seen in Exercise 3.6 that if $\mathcal{F}$ is a sheaf on a topological space X , then $\mathcal{F}(U)$ can be seen as a subset of $\prod_{\mathfrak{p} \in U} \mathcal{F}_{\mathfrak{p}}$. So if $U \subseteq \text{Spec } R$ is an open subset, we could try to define $\mathcal{O}(U) \subseteq \prod_{[\mathfrak{p}] \in U} R_{\mathfrak{p}}$. Hence any element in $\mathcal{O}(U)$ can be seen as a collection of elements $s_{\mathfrak{p}} \in R_{\mathfrak{p}}$ for $[\mathfrak{p}] \in U$. The condition on the principal open subset then forces on us the definition

$$\mathcal{O}(U) = \left\{ (s_{\mathfrak{p}}) \in \prod_{[\mathfrak{p}] \in U} R_{\mathfrak{p}} \mid \begin{array}{l} \text{there is an open cover of } U \text{ by sets } U_i \subseteq U \text{ and elements } g_i, f_i \in R \text{ such that} \\ f_i([\mathfrak{q}]) \neq 0 \text{ for all } [\mathfrak{q}] \in U_i \text{ and } s_{\mathfrak{q}} = \frac{g_i}{f_i} \text{ for all } [\mathfrak{q}] \in U_i \end{array} \right\}.$$

If $V \subseteq U$ is an open subset then we have restriction maps

$$\text{res}_V^U: \mathcal{O}(U) \rightarrow \mathcal{O}(V), \quad (s_{\mathfrak{p}})_{[\mathfrak{p}] \in U} \mapsto (s_{\mathfrak{p}})_{[\mathfrak{p}] \in V}$$

and these are clearly compatible, in the sense that if $W \subseteq V \subseteq U$ are open subsets, then $\text{res}_W^U = \text{res}_W^V \circ \text{res}_V^U$. We check that this has the properties that we want:

- *This is a sheaf of rings.* We see that being a sheaf of sets is fairly automatic: assume that $U = \bigcup U_i$ is an open cover of the open subset $U \subseteq \text{Spec } R$. If we have two elements $(s_{\mathfrak{p}}), (s'_{\mathfrak{p}}) \in \mathcal{O}(U)$ whose restriction to each U_i coincides, then $s_{\mathfrak{p}} = s'_{\mathfrak{p}}$ for each $[\mathfrak{p}] \in U$. Conversely, assume that we have elements $(s_{\mathfrak{p}}^{(i)}) \in \mathcal{O}(U_i)$ such that their restriction to each $U_i \cap U_j$ coincides. Then $s_{\mathfrak{p}}^{(i)} = s_{\mathfrak{p}}^{(j)}$ for each $\mathfrak{p} \in U_i \cap U_j$ and this means that there is $(s_{\mathfrak{p}}) \in \prod_{[\mathfrak{p}] \in U} R_{\mathfrak{p}}$ such that $s_{\mathfrak{p}}^{(i)} = s_{\mathfrak{p}}$ for each $[\mathfrak{p}] \in U_i$. Finally we need to check that $(s_{\mathfrak{p}}) \in \mathcal{O}(U)$ but we can just use the open cover given by the U_i .

Now we check that this is a sheaf of rings: take two elements $(s_{\mathfrak{p}}), (s'_{\mathfrak{p}}) \in \mathcal{O}(U)$. By definition around any point $[\mathfrak{p}] \in U$ we can find an open neighborhood $V \subseteq U$ and elements $g, f, g', f' \in R$ such that $f([\mathfrak{q}]) \neq 0, f'([\mathfrak{q}]) \neq 0$ for any $[\mathfrak{q}] \in V$ and $s_{\mathfrak{q}} = \frac{g}{f}$ and $s'_{\mathfrak{q}} = \frac{g'}{f'}$ for all $[\mathfrak{q}] \in V$. Then for any $[\mathfrak{q}] \in V$ we have that $(ff')([\mathfrak{q}]) \neq 0$ and

$$s_{\mathfrak{q}} + s'_{\mathfrak{q}} = \frac{g}{f} + \frac{g'}{f'} = \frac{gf' + g'f}{ff'}, \quad s_{\mathfrak{q}} \cdot s'_{\mathfrak{q}} = \frac{g}{f} \cdot \frac{g'}{f'} = \frac{gg'}{ff'}$$

and this proves that $\mathcal{O}(U)$ is a subring of $\prod_{[\mathfrak{p}] \in U} R_{\mathfrak{p}}$. It is also clear that the restriction maps are ring homomorphisms. Hence this is a sheaf of rings.

• *The description on principal open subsets:* Let $f \in R$. For each $[\mathfrak{p}] \in D(f)$ we have natural ring homomorphisms $R_f \rightarrow R_{\mathfrak{p}}, \frac{g}{f^n} \mapsto \frac{g}{f^n}$ so that this gives also natural ring homomorphisms

$$(4.3) \quad R_f \longrightarrow \mathcal{O}(D(f)), \quad \frac{g}{f^n} \mapsto \left(\frac{g}{f^n} \right)_{[\mathfrak{p}] \in D(f)}$$

We prove that (4.3) is injective: assume that $\frac{g}{f^n} = 0$ in $R_{\mathfrak{p}}$ for all $[\mathfrak{p}] \in D(f)$. This means that for any $[\mathfrak{p}] \in D(f)$ there is $s_{\mathfrak{p}} \in R \setminus \mathfrak{p}$ such that $s_{\mathfrak{p}} \cdot g = 0$. Consider then the ideal

$$\text{Ann}(g) = \{s \in R \mid s \cdot g = 0\}$$

The condition before means precisely that $V(\text{Ann}(g)) \cap D(f) = \emptyset$, which is equivalent to $V(\text{Ann}(g)) \subseteq V(f)$, which is then equivalent to $\sqrt{(f)} \subseteq \sqrt{\text{Ann}(g)}$. This means in particular that $f^n \cdot g = 0$ for a certain n , so that $\frac{g}{f^n} = 0$ in R_f .

We prove that (4.3) is surjective: assume we have $(s_{\mathfrak{p}}) \in \mathcal{O}(D(f))$. Then there is an open cover U_i of $D(f)$ and elements $g_i, f_i \in R$ with $f_i([\mathfrak{q}]) \neq 0$ for all $[\mathfrak{q}] \in U_i$ and $s_{\mathfrak{q}} = \frac{g_i}{f_i}$ for each $[\mathfrak{q}] \in U_i$. First we observe that $U_i \subseteq D(f_i)$ so that $D(f) = \bigcup D(f_i) \cap D(f) = \bigcup D(f_i f_i)$ and if we replace f_i by $f_i f_i$ we can assume that $U_i = D(f_i)$. By the quasi-compactness of $D(f)$ we can also assume that these are finitely many, so that $D(f) = D(f_1) \cup \dots \cup D(f_r)$ and there are $g_1, \dots, g_r \in R$ such that $s_{\mathfrak{q}} = \frac{g_i}{f_i}$ for each $[\mathfrak{q}] \in D(f_i)$. In particular, we know that

$$\frac{g_i}{f_i} = s_{\mathfrak{p}} = \frac{g_j}{f_j} \quad \text{in } R_{\mathfrak{p}} \text{ for each } \mathfrak{p} \in D(f_i) \cap D(f_j) = D(f_i f_j)$$

By the injectivity part that we have already proven, this means that $\frac{g_i}{f_i} = \frac{g_j}{f_j}$ in $R_{f_i f_j}$ meaning that there is $n_{ij} \in \mathbf{N}$ such that $(f_i f_j)^{n_{ij}} \cdot g_i f_j = (f_i f_j)^{n_{ij}} \cdot g_j f_i$ in R . Taking the maximum of all n_{ij} we show that there is $n \in \mathbf{N}$ such that $(f_i f_j)^n g_i f_j = (f_i f_j)^n g_j f_i$. Now observe that $D(f) = D(f_1^{n+1}) \cup \dots \cup D(f_r^{n+1})$ so that by Lemma 4.23 there are $h_1, \dots, h_r \in R$ and $N \in \mathbf{N}$ such that

$$f^N = h_1 f_1^{n+1} + \dots + h_r f_r^{n+1} \quad \text{in } R$$

Define now $g = h_1 f_1^n g_1 + \dots + h_r f_r^n g_r$: we claim that

$$\frac{g}{f^N} = \frac{g_i}{f_i} \quad \text{in } R_{f_i} \text{ for all } i = 1, \dots, r$$

which then implies that our original choice $(s_{\mathfrak{p}}) \in \mathcal{O}(D(f))$ is precisely the image of $\frac{g}{f^N}$ along the map of (4.3). We are left with proving the claim, and this is given by a computation:

$$\begin{aligned} f_i^n \cdot f_i g &= f_i^{n+1} g = h_1 (f_1 f_i)^n (f_i g_1) + \dots + h_r (f_r f_i)^n (f_i g_r) \\ &= h_1 (f_1 f_i)^n (f_1 g_i) + \dots + h_r (f_r f_i)^n (f_r g_i) \\ &= f_i^n g_i \cdot (h_1 f_1^{n+1} + \dots + h_r f_r^{n+1}) = f_i^n \cdot g_i f^N \end{aligned}$$

This completes the proof that the map (4.2) is an isomorphism. Taking $f = 1$, we get that the map $R \rightarrow \mathcal{O}(\text{Spec } R)$ is an isomorphism.

• *The description of the stalks:* Fix a point $[\mathfrak{p}] \in \operatorname{Spec} R$. We want to show that the following map is a ring isomorphism:

$$(4.4) \quad R_{\mathfrak{p}} \longrightarrow \mathcal{O}_{[\mathfrak{p}]}, \quad \frac{g}{f} \mapsto \left[\left(\frac{g}{f} \right)_{[\mathfrak{q}] \in D(f)} \right]$$

where on the right we mean the germ in $\mathcal{O}_{[\mathfrak{p}]}$ given by $(g/f)_{[\mathfrak{q}] \in D(f)}$ in $\mathcal{O}(D(f))$. Notice that this is the element corresponding to $g/f \in R_f$ via the isomorphism $R_f \rightarrow \mathcal{O}(D(f))$ of (4.3). We first prove that the map is well-defined: if $\frac{g}{f} = \frac{g'}{f'}$ in $R_{\mathfrak{p}}$ then there is $h \in R \setminus \mathfrak{p}$ such that $hgf' = hg'f$. This implies that $\frac{g}{f} = \frac{g'}{f'}$ in $R_{\mathfrak{q}}$ for any $\mathfrak{q} \in D(ff'h)$ and this shows that the map is well-defined. It is also immediate that it is a ring homomorphism.

By construction, the image of (4.4) is precisely given by the germs of sections in $\mathcal{O}(D(f))$ for principal open subsets $D(f)$ containing $[\mathfrak{p}]$. Since the principal open sets are a basis of the Zariski topology, every element in $\mathcal{O}_{[\mathfrak{p}]}$ is the germ of such a section. This means that the map (4.4) is surjective.

For the injectivity, if $\frac{g}{f} \in R_{\mathfrak{p}}$ is in the kernel of (4.4), this means that there is an open set $D(h) \subseteq D(f)$ such that $\frac{g}{f} = 0$ in $\mathcal{O}(D(f))$. Since the map $R_h \rightarrow \mathcal{O}(D(f))$ of (4.3) is an isomorphism, this means that $\frac{g}{f} = 0$ in R_h and then $\frac{g}{h} = 0$ in $R_{\mathfrak{p}}$ as well. $\square$

Definition 4.28 (Affine scheme). The locally ringed space $(\operatorname{Spec} R, \mathcal{O}_{\operatorname{Spec} R})$ is called the affine scheme associated to R .

We will usually write $\operatorname{Spec} R$ to denote the affine scheme, without mentioning the sheaf of rings explicitly. Moreover we will also write $\mathcal{O}$ instead of $\mathcal{O}_{\operatorname{Spec} R}$. If $U \subseteq \operatorname{Spec} R$ is open, we will sometimes denote the elements in $\mathcal{O}_{\operatorname{Spec} R}(U)$ as *regular functions* on U , but we have to keep in mind that they are not functions in the usual sense.

Example 4.29 (Affine space over a ring). If R is a ring, we denote

$$\mathbf{A}_R^n := \operatorname{Spec} R[x_1, \dots, x_n]$$

and we call it the affine n -space over the ring R .

Exercise 4.30. If $R = \mathbf{C}, \mathbf{C}[x], \mathbf{Z}$ determine $\mathcal{O}(U)$ for any open subset $U \subseteq \operatorname{Spec} R$. $\diamond$

Exercise 4.31. Let k be any field, let $0 = [(x, y)] \in \mathbf{A}_k^2$ be the point corresponding to the maximal ideal (x, y) . This is a closed point so that its complement $\mathbf{A}_k^2 \setminus \{0\}$ is open. Show that the restriction map

$$\mathcal{O}(\mathbf{A}_k^2) \longrightarrow \mathcal{O}(\mathbf{A}_k^2 \setminus \{0\})$$

is an isomorphism, so that $\mathcal{O}(\mathbf{A}^2 \setminus \{0\}) \cong k[x, y]$. $\diamond$

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