



## Sheet 1

If you want feedback on your solutions, you should upload them on URM by October 27 at 10:00.

Please write your name and matriculation number in any sheet that you hand in. If you are handing in the solutions as a group it is enough for one person in the group to upload the solutions. We might not have the time to correct all your solutions, so it would be useful if you could mark the solutions on which you particularly want feedback.

**Exercise 1.1.** Let  $X$  be a topological space and  $\mathcal{F}$  a sheaf of sets on  $X$ . If  $U \subseteq X$  is an open subset and  $s \in \mathcal{F}(U)$  a section, we denote by  $s_p \in \mathcal{F}_p$  its germ at a point  $p \in U$ .

- (i) If  $U \subseteq X$  is an open subset and  $s, t \in \mathcal{F}(U)$  two sections, prove that the set  $\{p \in U \mid s_p = t_p\}$  is open in  $U$ .

Assume now that  $(X, \mathcal{O}_X)$  is a locally ringed space. Recall that if  $U \subseteq X$  is an open subset,  $s \in \mathcal{O}_X(U)$  a section and  $p \in U$  a point, the value  $s(p) \in \kappa(p)$  of  $s$  at  $p$  is defined to be the image of  $s$  along the composition

$$\mathcal{O}_X(U) \rightarrow \mathcal{O}_{X,p} \rightarrow \mathcal{O}_{X,p}/\mathfrak{m}_p = \kappa(p), \quad s \mapsto s_p \mapsto s(p)$$

- (ii) If  $U \subseteq X$  is an open subset and  $s, t \in \mathcal{O}_X(U)$  two sections, prove that the set  $\{p \in U \mid s(p) = t(p)\}$  is closed in  $U$ .  $\diamond$

Let  $X$  be a topological space and  $\varphi: \mathcal{F} \rightarrow \mathcal{G}$  a morphism of sheaves of sets on  $X$ . We define  $\varphi$  to be *injective* (resp. *surjective*) if the induced map on stalks  $\varphi_p: \mathcal{F}_p \rightarrow \mathcal{G}_p$  is *injective* (resp. *surjective*) for any  $p \in X$ .

**Exercise 1.2.** Let  $X$  be a topological space and  $\varphi: \mathcal{F} \rightarrow \mathcal{G}$  a morphism of sheaves of sets on  $X$ . Show that the following conditions are equivalent:

- (a)  $\varphi$  is injective.
- (b) for each open subset  $U \subseteq X$ , a section  $s \in \mathcal{F}(U)$  is determined by its image via  $\varphi$ . More precisely, this means that the map  $\varphi_U: \mathcal{F}(U) \rightarrow \mathcal{G}(U)$  is injective for each open subset  $U \subseteq X$ .
- (c)  $\varphi$  is a monomorphism in the categorical sense: if  $\psi, \psi': \mathcal{H} \rightarrow \mathcal{F}$  are morphism of sheaves such that  $\varphi \circ \psi = \varphi \circ \psi'$ , then  $\psi = \psi'$ .

Show then that the following conditions are equivalent:

- (d)  $\varphi$  is surjective.
- (e) for each open subset  $U \subseteq X$  a section  $t \in \mathcal{G}(U)$  is locally the image via  $\varphi$  of a section of  $\mathcal{F}$ . More precisely, this means there is an open cover  $U = \bigcup U_i$  of  $U$  and  $s_i \in \mathcal{F}(U_i)$  such that  $\varphi_{U_i}(s_i) = t|_{U_i}$ .
- (f)  $\varphi$  is an epimorphism in the categorical sense: if  $\psi, \psi': \mathcal{G} \rightarrow \mathcal{H}$  are morphism of sheaves such that  $\psi \circ \varphi = \psi' \circ \varphi$ , then  $\psi = \psi'$ .  $\diamond$

In particular, if a morphism of sheaves  $\varphi: \mathcal{F} \rightarrow \mathcal{G}$  is surjective, it **does not** mean in general that  $\varphi_U: \mathcal{F}(U) \rightarrow \mathcal{G}(U)$  is surjective for each  $U \subseteq X$ . If you know some basic complex analysis, the following exercise gives the best example for this phenomenon:

**Exercise 1.3.** Consider  $X = \mathbf{C}$  with the usual Euclidean topology and the two sheaves  $\mathcal{O}_X^{an}$  and  $\mathcal{O}_X^{an,\times}$  given on any open subset  $U \subseteq X$  by

$$\mathcal{O}_X^{an}(U) = \{f: X \rightarrow \mathbf{C} \mid f \text{ holomorphic}\}, \mathcal{O}_X^{an,\times}(U) = \{f: X \rightarrow \mathbf{C}^\times \mid f \text{ holomorphic}\}$$

Consider for any open subset  $U \subseteq X$  the exponential map

$$\exp_U: \mathcal{O}_X^{an}(U) \rightarrow \mathcal{O}_X^{an,\times}(U), \quad f \mapsto \exp(f).$$

- (i) Show that this defines a morphism of sheaves of abelian groups  $\exp: \mathcal{O}_X^{an} \rightarrow \mathcal{O}_X^{an,\times}$ .
- (ii) Show that this morphism of sheaves is surjective.
- (iii) Find an open subset  $U \subseteq X$  such that  $\exp_U: \mathcal{O}_X^{an}(U) \rightarrow \mathcal{O}_X^{an,\times}(U)$  is not surjective.  $\diamond$

**Exercise 1.4.** Let  $X$  be a topological space and  $\varphi: \mathcal{F} \rightarrow \mathcal{G}$  a morphism of sheaves of sets/rings/abelian groups/... on  $X$ . Recall that  $\varphi$  is an isomorphism if there is another morphism of sheaves of sets/rings/abelian groups/...  $\psi: \mathcal{G} \rightarrow \mathcal{F}$  such that  $\varphi \circ \psi = \text{id}_{\mathcal{G}}, \psi \circ \varphi = \text{id}_{\mathcal{F}}$ . Show that the following are equivalent:

- (a)  $\varphi$  is an isomorphism.
- (b)  $\varphi$  is injective and surjective.
- (c)  $\varphi_U: \mathcal{F}(U) \rightarrow \mathcal{G}(U)$  is an isomorphism of sets/rings/abelian groups/... for any open subset  $U \subseteq X$ .  $\diamond$