



Sheet 2

If you want feedback on your solutions, you should upload them on URM by November 3 at 10:00.

Please write your name and matriculation number in any sheet that you hand in. If you are handing in the solutions as a group it is enough for one person in the group to upload the solutions. We might not have the time to correct all your solutions, so it would be useful if you could mark the solutions on which you particularly want feedback.

Recall that a topological space X is *connected* if, whenever we write $X = U_1 \cup U_2$ as the union of two non-empty open (or closed) subsets, then $U_1 \cap U_2 \neq \emptyset$. Hence X is *disconnected* if there are two non-empty open (or closed) subsets $U_1, U_2 \subseteq X$ such that $X = U_1 \cup U_2$ and $U_1 \cap U_2 = \emptyset$.

Exercise 2.1. Let R be a ring. Show that the following are equivalent

- (a) R has a non-trivial idempotent element: this means that there is $x \in R \setminus \{0, 1\}$ such that $x^2 = x$.
- (b) There are two proper ideals $I_1, I_2 \subseteq R$, such that $I_1 \cap I_2 = 0$ and that are coprime, meaning that $I_1 + I_2 = R$.
- (c) R is isomorphic to a product of two non-zero rings $R \cong R_1 \times R_2$.
- (d) $\text{Spec } R$ is disconnected. ◇

Exercise 2.2. If R is a local ring, show that $\text{Spec } R$ is connected. ◇

Now let $\phi: S \rightarrow R$ be a ring homomorphism. In the lectures we define an associated map

$$\tilde{\phi}: \text{Spec } R \rightarrow \text{Spec } S, \quad [\mathfrak{p}] \mapsto [\phi^{-1}(\mathfrak{p})]$$

and we show that this is well-defined and continuous (prove it yourself if we haven't seen it in the lectures already).

Exercise 2.3. Let $\phi: S \rightarrow R$ be a ring homomorphism and let $\tilde{\phi}: \text{Spec } R \rightarrow \text{Spec } S$ be the associated map.

- (i) If $I \subseteq R$ is an ideal, show that the closure of $\tilde{\phi}(V(I))$ in $\text{Spec } S$ is given by

$$\overline{\tilde{\phi}(V(I))} = V(\phi^{-1}(I)).$$

- (ii) Show that the map is dominant, meaning that the image is dense, if and only if $\sqrt{\text{Ker } \phi} = \sqrt{(0)}$.
- (iii) If S is reduced, show that $\tilde{\phi}$ is dominant if and only if ϕ is injective. ◇

Turn the sheet.

For the last exercise, recall that a continuous map $f: X \rightarrow Y$ of topological spaces is *closed* (resp. *open*) if the image of a closed (resp. open) subset in X is a closed (resp. open) subset in Y . It is always true that if f is bijective, then f is a homeomorphism if and only if it is open or closed.

Exercise 2.4. Let R be a ring.

- (i) Let $I \subseteq R$ be an ideal, and $\pi: R \rightarrow R/I$ the projection. Show that the map $\tilde{\pi}: \operatorname{Spec} R/I \rightarrow \operatorname{Spec} R$ is closed, injective and with image equal to $V(I)$. Deduce that $\operatorname{Spec} R/I \rightarrow V(I)$ is a homeomorphism.
- (ii) Let $f \in R$ be an element and $\alpha_f: R \rightarrow R_f$ the canonical map $a \mapsto \frac{a}{1}$. Show that the map $\tilde{\alpha}_f: \operatorname{Spec} R_f \rightarrow \operatorname{Spec} R$ is open, injective and with image equal to $D(f)$. Deduce that $\operatorname{Spec} R_f \rightarrow D(f)$ is a homeomorphism.
- (iii) Let $[\mathfrak{p}] \in \operatorname{Spec} R$ be a prime ideal and $\alpha_{\mathfrak{p}}: R \rightarrow R_{\mathfrak{p}}$ the canonical map $a \mapsto \frac{a}{1}$. Show that the map $\tilde{\alpha}_{\mathfrak{p}}: \operatorname{Spec} R_{\mathfrak{p}} \rightarrow \operatorname{Spec} R$ is injective and that its image is the intersection of all open sets in $\operatorname{Spec} R$ that contain $[\mathfrak{p}]$. Show that the map $\tilde{\alpha}_{\mathfrak{p}}$ is not open or closed in general. $\diamond$