Kdv-solitions: A tropical view

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Recall the KdV-equation

$$4u_t = u_{xxx} + 6uu_x. \tag{1}$$

Our goal is to study n-solition equations i.e n distinct traveling waves which are almost unchanged after interaction. See for example this video: `https://www.youtube.com/watch?v=wEbYELtGZwI`. One can think of the waves as particles, the so called solitons. Tropical geometry will turn out to be helpful in describing the solitons at all times. Here is the cookbook solution (based on the Direct method of Hirota). For $0 < p_1 < \ldots < p_M$ and $c_1, \ldots, c_M \in \mathbb{R}$ set $\phi_j = e^{p_j x + p_j^3 t + c_j}$

$$\phi_j = e^{p_j x + p_j^3 t + c_j} - e^{-p_j x - p_j^3 t - c_j},\ j = 1, \ldots, M.$$

Then we define the τ-function

$$\tau(x,t) = \mathrm{Wr}(\phi_1, \ldots, \phi_M) = \det(\partial_x^{i-1}\phi_j)_{i,j=1,\ldots,M}.$$

Then $u = 2\partial_x^2 \log\tau$ solves the KdV-equation. For tropicalizing τ we rewrite this function as a sum of exponential functions.

$$\begin{aligned}
\tau &= \det\left(p_j^{i-1}(e^{p_j x + p_j^3 t + c_j} - (-1)^{j-1} e^{-p_j x - p_j^3 t - c_j})\right) \text{ Multilinearity} \\
&= \sum_{A=(\alpha_1,\ldots,\alpha_M)\in\{1,-1\}^M} \det\left(\alpha_j(\alpha_j p_j)^{i-1} e^{\alpha_j(p_j x + p_j^3 t + c_j)}\right) \\
&= \sum_A e^{\sum_j \alpha_j(p_j x + p_j^3 t + c_j)} |\underbrace{\Delta(\alpha_1 p_1, \ldots, \alpha_M p_M)}_{\text{Vandermonde determinant}}| \\
&= \sum_A e^{\Theta_A}.
\end{aligned}$$

Here we have put $\theta_j = p_j x + p_j^3 t + c_j$,$\delta_A = \log|\Delta(\alpha_1 p_1, \ldots, \alpha_M p_M)|$ and $\Theta_A = \sum_{j=1}^M \alpha_j\theta_j + \delta_A$. The heuristic idea is as follows. We have very roughly $\log\tau = \max\{\Theta_A : A \in \{1,-1\}^M\}$. Then the second logarithmic derivative is almost zero is close to zero when the maximum is assumed only once, because Θ_A is linear in x and t. One way to make this more precise is Maslov dequantisation. One degenerates τ as follows.

$$\lim_{\epsilon\to 0} \epsilon \log\left(\sum_A e^{\Theta_A/\epsilon}\right) = \max\{\Theta_A : A \in \{1,-1\}^M\}.$$

To apply the notation we have learned the last time let us write the "logarithm" of τ as a tropical polynomial

$$\bigoplus_{A\in\{1,-1\}^M}\left(x^{\odot\sum\alpha_j p_j}\odot t^{\odot\sum_j \alpha_j p_j^3}\odot(\sum_j \alpha_j c_j)\odot\delta_A\right). \tag{2}$$

This is not your usual tropical polynomial because the exponents are allowed to be irrational. But the approach via the dual subdivision should work just as nicely in this case. The newton polytope is obtained by projecting a hypercube with vertices $(\alpha_j)_{j=1,\dots,M}$ in $\mathbb{R}^M$ via the linear map $\pi:\mathbb{R}^M\to\mathbb{R}^2$ with matrix

$$\begin{pmatrix} p_1 & p_2 & \dots & p_M\\ p_1^3 & p_2^3 & \dots & p_M^3\end{pmatrix}. \tag{3}$$

By the way Dimakis and Müller-Hoissen call the monomial in (2) phases. We have learned from Hannah that the tropical curve W defined by our polynomial is piecewise linear and its line seqments correspond to points where two phases are equal. We label the line seqments by tuples (A,B), where A,B are in $\{1,-1\}^M$. Ends of the tropical curve correspond to the lines at the boundary of the Newton polygon. A necessary condition for (A,B) to give an end is that $A-B=e_i=(0,\dots,0,1,0,\dots,0)$ holds for some i. We can do better and find explicitely the corners of the Newton polytope. We identify them with the following elements of $\{1,-1\}^M$.

$$A_j = (\overbrace{-1,\dots,-1}^{j\text{ times}},1,\dots,1)$$
$$B_j = (\overbrace{1,\dots,1}^{j\text{ times}},-1,\dots,-1),\ j=0,\dots,M.$$

Proposition 0.1. *The corners of the newton polytope of (2) are the points $\pi(A_j)$ and $\pi(B_j)$ for $j=1,\dots,N$.*

Proof. We just show that $\pi(A_j)$ is an extremal point, the proof for $\pi(B_j)$ being almost the same. Pick q between p_j and p_{j+1}. Denote the vector (q,q^3) by n. We have

$$n\cdot\pi(A_j)=\sum_{i=1}^{j} p_i q\overbrace{(q^2-p_i^2)}^{>0}+\sum_{i=j+1}^{M} p_i q\overbrace{(p_i^2-q^2)}^{>0}.$$

If we flip any signs in the entries of A_j some summands become smaller, while others remain the same. Hence $n\cdot v$ is maximized for $v=\pi(A_j)$. □

A similar proof shows that B_{j-1} and B_j respectively A_{j-1} and A_j are connected by a face of the Newton polytope. They corresponds to the unbounded edges of the tropical curve. They can be interpreted as the incoming and outgoing trajectory of the j-th soliton.

More generally a bounded edge corresponding to two vertices of the hypercube only differing in the j-th coordinate can be interpreted as a manifestation of the n-th solition. Not all bounded edges of the tropical curve are of this form. The incoming trajectory and the outgoing trajectory of the n-th soliton are parallel but they differ by a translation called the phase shift.