The Korteweg-de Vries equation and theta functions

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We consider the Korteweg-de Vries (KdV) partial differential equation

$$4u_t - 6uu_x - u_{xxx} = 0 \tag{1}$$

The function $u = u(x,t)$ models the height of one-dimensional waves in shallow water.

1 Translation waves solutions and plane cubics

We look for translation waves solutions of the form

$$u(x,t) = 2 \cdot v(x + a \cdot t)$$

for a constant a and a function $v = v(z)$ that we take to be as nice as possible, meaning analytic. Imposing that this is a solution to the KdV we get

$$8a \cdot v' - 24vv' - 2v''' = 0$$

This is a differential equation for the function v. Integrating this we get

$$8av - 12v^2 - 2v'' + b = 0$$

where b is an integration constant. We can multiply this equation by v' to obtain

$$8avv' - 12v^2v' - 2v''v' + bv' = 0$$

and we can integrate this again to obtain

$$4av^2 - 4v^3 - (v')^2 + bv + c = 0$$

where c is again an integration constant. In conclusion, we proved the following

Lemma 1.1. *Let v be an analytic function which satisfies a differential equation of the form*

$$(v')^2 = -4 \cdot v^3 + 4a \cdot v^2 + b \cdot v + c$$

for certain constants a, b, c. Then the function $u(x,t) = 2v(x + at)$ is a solution to the KdV equation.

In geometric terms, this leads naturally to the plane cubic curve

$$E = \{(x, y) \in \mathbb{C}^2 \,|\, y^2 = -4 \cdot x^3 + 4a \cdot x^2 + b \cdot x + c\}$$

and we are looking for an analytic function v such that the map

$$(v, v')\colon \mathbb{C} \longrightarrow E, \qquad z \mapsto (v(z), v'(z))$$

ends up into E. A plane cubic curve in the form $\{y^2 = f(x)\}$, where $f(x)$ is a polynomial of degree three is said to be in Weierstraß form, and Weierstraß actually found a function that satisfies the differential equation of Lemma 1.1. This function is known as the Weierstraß $\wp$-function. However, here we will consider it through the point of view of the theta function.

2 Complex tori and theta functions

Let $\mathcal{H} = \{\tau \in \mathbb{C} \,|\, \operatorname{Im} \tau > 0\}$ be the complex upper-half plane, consisting of complex numbers with positive imaginary part.

Definition 2.1 (Theta function)**.** The theta function is

$$\theta\colon \mathbb{C} \times \mathcal{H} \to \mathbb{C}, \qquad \theta(z, \tau) = \sum_{n \in \mathbb{Z}} \exp\left(\pi i n^2 \tau + 2\pi i n z\right)$$

Remark 2.2. Since we are imposing that $\operatorname{Im} \tau > 0$, the series in the definition converges uniformly on compact subsets, so that the theta function is a well-defined holomorphic function.

The most important property of the theta function is its quasiperiodicity:

Proposition 2.3 (Quasiperiodicity)**.** *Let $m, n \in \mathbb{Z}$. Then*

$$\theta\left(z + m + \tau n, \tau\right) = \exp\left(-\pi i n^2 \tau - 2\pi i n z\right) \cdot \theta(z, \tau)$$

Proof. We can compute

$$\begin{aligned}
\theta(z + m + \tau n, \tau) &= \sum_{h \in \mathbb{Z}} \exp\left(\pi i h^2 \tau + 2\pi i h (z + m + n\tau)\right) \\
&= \sum_{h \in \mathbb{Z}} \exp\left(\pi i h^2 \tau + 2\pi i h n \tau + 2\pi i h z\right) \exp(2\pi i m h) \\
&= \sum_{h \in \mathbb{Z}} \exp\left(\pi i (h + n)^2 \tau - \pi i n^2 \tau + 2\pi i (n + h) z - 2\pi n z\right) \\
&= \exp\left(-\pi i n^2 \tau - 2\pi i n z\right) \cdot \sum_{h \in \mathbb{Z}} \exp\left(\pi i (h + n)^2 \tau + 2\pi i (n + h) z\right) \\
&= \exp\left(-\pi i n^2 \tau - 2\pi i n z\right) \cdot \theta(z, \tau)
\end{aligned}$$

□

This quasiperiodicity is interpreted geometrically through the action of the lattice $\Lambda_\tau = \{m + \tau n \,|\, m, n \in \mathbb{Z}\}$ on the complex plane $\mathbb{C}$. One can see that the quotient $\mathbb{C}/\Lambda_\tau$ is topologically equivalent to a real torus. Since it has also a complex structure inherited from $\mathbb{C}$ it is also called a complex torus. Before looking at some consequences of the quasiperiodicity, let's see another useful property of the theta function

Lemma 2.4 (Parity)**.** *The theta function is even, meaning that*

$$\theta(-z, \tau) = \theta(z, \tau)$$

Proof. We can simply compute

$$\theta(-z,\tau) = \sum_{n\in\mathbb{Z}} \exp\left(\pi i n^2 \tau - 2\pi i n z\right) = \sum_{n\in\mathbb{Z}} \exp\left(\pi(-n)^2\tau + 2\pi i(-n)z\right) = \theta(z,\tau)$$

□

One immediate consequence of the quasiperiodicity is on the logarithmic derivatives of the theta function

Lemma 2.5. *For any $n, m \in \mathbb{Z}$ it holds that*

$$\begin{aligned}
\frac{\partial \log\theta}{\partial z}(z + m + \tau n, \tau) &= -2\pi i n + \frac{\partial \log\theta}{\partial z}(z,\tau), \\
\frac{\partial^k \log\theta}{\partial z^k}(z + m + \tau n, \tau) &= \frac{\partial^k \log\theta}{\partial z^k}(z,\tau) \qquad \textit{for all } k \geq 2
\end{aligned}$$

Proof. Just take the logarithm in the quasiperiodicity and then the successive derivatives. □

With this we can identify the zeroes of the theta function

Proposition 2.6. *For each fixed $\tau \in \mathcal{H}$ the zeroes of the corresponding theta function are*

$$\{z \,|\, \theta(z,\tau) = 0\} = \left\{\frac{1}{2} + \frac{1}{2}\tau\right\} + \Lambda_\tau$$

furthermore, these are all simple zeroes.

Proof. The quasiperiodicity shows that all the zeroes are Λ_τ invariant. We are now going to show that, up to translation by Λ_τ, the theta function has an unique zero, which is moreover a simple zero. To do so, draw a fundamental parallelogram P for Λ_τ whose sides do not contain any zero. Then we want to show that there is a unique zero inside this fundamental parallelogram. The Residue Theorem applied to the logarithmic derivative shows that

$$\#\{\text{ zeroes of } \theta \text{ inside } P\} = \frac{1}{2\pi i}\int_{\partial P} \frac{\partial \log\theta}{\partial z} dz$$

however, the first relation of Lemma 2.5 shows that the integral on the right is exactly $2\pi i$. Hence, θ has only one zero inside P, counted with multiplicity. To conclude, we need to prove that $z_0 = \frac{1}{2} + \frac{1}{2}\tau$ is a zero of theta. We use a trick following Jacobi: consider the function

$$\theta_{11}(z,\tau) = \exp\left(\frac{1}{4}\pi i\tau + \pi i\left(z + \frac{1}{2}\right)\right) \cdot \theta\left(z + \frac{1}{2} + \frac{1}{2}\tau, \tau\right)$$

We claim that $\theta_{11}(z,\tau)$ is odd, meaning that $\theta_{11}(-z,\tau) = -\theta_{11}(z,\tau)$. If this is true, then $\theta_{11}(0,\tau) = 0$, meaning that $\theta\left(\frac{1}{2} + \frac{1}{2}\tau, \tau\right) = 0$. To prove that θ_{11} is odd we compute

$$\begin{aligned}
\theta_{11}(-z,\tau) &= \exp\left(\frac{1}{4}\pi i\tau + \pi i\left(-z + \frac{1}{2}\right)\right) \cdot \theta\left(-z + \frac{1}{2} + \frac{1}{2}\tau, \tau\right) \\
[\theta \text{ is even }] &= \exp\left(\frac{1}{4}\pi i\tau + \pi i\left(-z + \frac{1}{2}\right)\right) \cdot \theta\left(z - \frac{1}{2} - \frac{1}{2}\tau, \tau\right) \\
&= \exp\left(\frac{1}{4}\pi i\tau + \pi i\left(-z + \frac{1}{2}\right)\right) \cdot \theta\left(z + \frac{1}{2} + \frac{1}{2}\tau - 1 - \tau, \tau\right) \\
[\text{quasiperiodicity }] &= \exp\left(\frac{1}{4}\pi i\tau + \pi i\left(-z + \frac{1}{2}\right)\right) \cdot \exp\left(-\pi i\tau + 2\pi i\left(z + \frac{1}{2} + \frac{1}{2}\tau\right)\right) \\
&\cdot \theta\left(z + \frac{1}{2} + \frac{1}{2}\tau, \tau\right) \\
&= \exp(\pi i) \cdot \exp\left(\frac{1}{4}\pi i\tau + \pi i\left(z + \frac{1}{2}\right)\right) \cdot \theta\left(z + \frac{1}{2} + \frac{1}{2}\tau, \tau\right) \\
&= -\theta_{11}(z,\tau).
\end{aligned}$$

Remark: in the previous reasoning we did not need the factor $\exp\left(\frac{1}{4}\pi i\tau\right)$, but it is there for another reason (that we do not care about today) and we kept it for notational consistence. □

As an immediate consequence of this, we get

Corollary 2.7. The meromorphic functions $\frac{\partial^2 \log\theta}{\partial z^2}, \frac{\partial^3\theta}{\partial z^3}, \left(\frac{\partial^2 \log\theta}{\partial z^2}\right)^2$ are Λ_τ periodic, and, up to translations by Λ_τ they have a unique pole at $z_0 = \frac{1}{2} + \frac{1}{2}\tau$. Furthermore, this point is a pole of order $2, 3, 4$ respectively.

Proof. If θ has a simple zero at z_0, the first logarithmic derivative has a simple pole at z_0, and then the rest follows by taking higher derivatives. □

We also need some more complex analysis

Lemma 2.8. *Any Λ_τ-periodic meromorphic function that, up to Λ_τ-translations, has at most a simple pole is constant.*

Proof. Suppose that f is a meromorphic function such that $f(z + m + \tau n) = f(z)$ for all $m, n \in \mathbb{Z}$. Assume that, up to translation, f has at most one simple pole x. We can put x into a fundamental parallelogram P and then the Residue Theorem tells us that

$$\operatorname{Res}(f,x) = \frac{1}{2\pi i}\int_{\partial P} f(z)dz$$

Since f is Λ_τ-periodic, the integral on the right is zero, so that the function has no residues at f. But this means that x is not a pole of f. Thus, f must actually be holomorphic. Since $f\colon \mathbb{C} \to \mathbb{C}$ is Λ_τ-invariant, it factors through a holomorphic function $\overline{f}\colon \mathbb{C}/\Lambda_\tau \to \mathbb{C}$ but since the space $\mathbb{C}/\Lambda_\tau$ is a complex torus, it is compact, and its image is bounded. Then f is a holomorphic bounded function, hence it is constant. □

With this we can give the result that we were looking for:

Theorem 2.9. *There exist $a, b, c \in \mathbb{C}$ such that*

$$\left(\frac{\partial^3 \log\theta}{\partial z^3}\right)^2 = -4\left(\frac{\partial^2 \log\theta}{\partial z^2}\right)^3 + 4a\cdot\left(\frac{\partial^2 \log\theta}{\partial z^2}\right)^2 + b\cdot\left(\frac{\partial^2 \log\theta}{\partial z^2}\right) + c$$

Proof. A computation, for example on a computer, shows that for an arbitrary function $f(z)$ it holds that

$$\left(\frac{\partial^3 \log f}{\partial z^3}\right)^2 + 4\left(\frac{\partial^2 \log f}{\partial z^2}\right)^3 = \frac{4f'''(z)f'(z)^3}{f(z)^4} - \frac{3f'(z)^2 f''(z)^2)}{f(z)^4} + \frac{4f''(z)^3}{f(z)^3} - \frac{6f^(3)(z)f'(z)f''(z))}{f(z)^3} + \frac{f'''(z)^2}{f(z)^2}$$

The precise form is not important, but what is important is the highest power of $f(z)$ appearing in a denominator is 4. Hence, if we apply this to the theta function, we see that the expression

$$\left(\frac{\partial^3 \log\theta}{\partial z^3}\right)^2 + 4\left(\frac{\partial^2 \log\theta}{\partial z^2}\right)^3$$

can be written as the quotient of an holomorphic function by $\theta(z,\tau)^4$. Hence, the full expression has a pole of order at most 4 at $z_0 = \frac{1}{2} + \frac{1}{2}\tau$. We can subtract an appropriate multiple of $\left(\frac{\partial^2 \log\theta}{\log z^2}\right)^2$ and obtain a function with a pole at z_0 of order at most 3, we can then subtract an appropriate multiple of $\frac{\partial^3 \log\theta}{\log z^3}$ to obtain a function with a pole of order at most 2 and then we can subtract an appropriate multiple of $\frac{\partial^2 \log\theta}{\log z^2}$ to obtain a function with a pole of order at most 1. To, summarize, there are $a, b, d \in \mathbb{C}$ such that the Λ_τ-periodic function

$$\left(\frac{\partial^3 \log\theta}{\partial z^3}\right)^2 + 4\left(\frac{\partial^2 \log\theta}{\partial z^2}\right)^3 - 4a\cdot\left(\frac{\partial^2 \log\theta}{\partial z^2}\right)^2 - d\cdot\left(\frac{\partial^3 \log\theta}{\partial z^3}\right) - b\cdot\left(\frac{\partial^2 \log\theta}{\partial z^2}\right)$$

has a pole of order at most 1 at z_0 and, up to Λ_τ-translation, nowhere else. Then Lemma 2.8 shows that the function must be constant. Hence there is $c \in \mathbb{C}$ such that

$$\left(\frac{\partial^3 \log\theta}{\partial z^3}\right)^2 = -4\left(\frac{\partial^2 \log\theta}{\partial z^2}\right)^3 + 4a\cdot\left(\frac{\partial^2 \log\theta}{\partial z^2}\right)^2 - d\cdot\left(\frac{\partial^3 \log\theta}{\partial z^3}\right) + b\cdot\left(\frac{\partial^2 \log\theta}{\partial z^2}\right) + c$$

To conclude, we just need to show that $d = 0$. But this is true because the function $\frac{\partial^3 \log\theta}{\partial z^3}$ is odd, while all other functions appearing are even. □

This way, we get a solution to the KdV equation

Corollary 2.10. *With the same notation as in Theorem 2.9, the function*

$$u(x,t) = 2 \cdot \frac{\partial^2 \log \theta}{\partial z^2}(x + a \cdot t)$$

is a solution to the KdV equation

These solutions are usually called *quasiperiodic.*

3 Degenerations and soliton solutions

We can get other solutions to the KdV by degenerating the theta function:

Lemma 3.1. *Let $t \in \mathbb{R}$ be a positive real number. Then*

$$\lim_{t \to +\infty} \theta\left(z - \frac{1}{2}it, it\right) = 1 + \exp(2\pi i z)$$

Proof. We just write everything explicitly:

$$\begin{aligned} \theta\left(z - \frac{1}{2}it, it\right) &= \sum_{n \in \mathbb{Z}} \exp\left(\pi i n^2 \cdot it + 2\pi i n\left(z - \frac{1}{2}it\right)\right) \\ &= \sum_{n \in \mathbb{Z}} \exp\left(-\pi(n^2 - n)t\right) \cdot \exp(2\pi i n z). \end{aligned}$$

We see that $n^2 - n \geq 0$ for all $n \in \mathbb{Z}$ and when $n^2 - n > 0$ the term $\exp\left(-\pi(n^2 - n)t\right) \cdot \exp(2\pi i n z)$ goes to zero as $t \to +\infty$. The only terms surviving are those for which $n^2 - n = 0$: this means precisely $n = 0, 1$ so that the limit is exactly what we want. □

This way we have obtained a sort of degenerate theta function

$$\hat{\theta}(z) = 1 + \exp(2\pi i z)$$

Do we still get a solution of the KdV equation out of this? One can compute that

$$\left(\frac{\partial^3 \log \hat{\theta}}{\partial z^3}\right)^2 + 4\left(\frac{\partial^2 \log \hat{\theta}}{\partial z^2}\right)^3 = (2\pi i)^2 \cdot \left(\frac{\partial^2 \log \hat{\theta}}{\partial z^2}\right)^2$$

so that the function

$$u(x,t) = 2\frac{\partial^2 \log}{\partial z^2}\hat{\theta}(x + \pi^2 t)$$

is a solution to the KdV equation thanks to Lemma 1.1. This is what one could call a soliton solution to the KdV equation. Geometrically, what is happening is that the curve $y^2 = -4x^3 - 4\pi^2 \cdot x^2$ is now a singular cubic curve, with a nodal singularity at $(x, y) = (0, 0)$.