Convex Geometry

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Introduction

The lecture notes provided by Hannah Markwig have been set in LaTeXby Monica Janders. If you find any mistakes, please contact me at monica.janders@student.uni-tuebingen.de.

Convex geometry deals with polytopes, cones, polyhedra,..

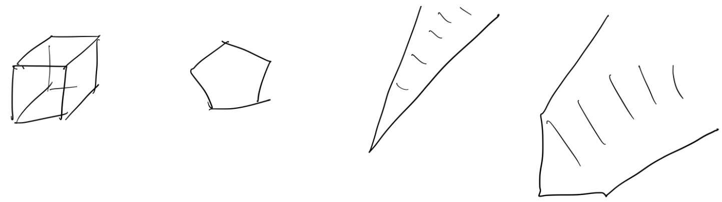


Abbildung 1: Polytopes, cones, polyhedra,...[1]

It combines geometry, combinatorics (and (linear) algebra).
It trains your visual thinking (in particular spatial imagination an beyond).

We will start with **Pick's Formula**. We look at a lattice, like on a piece of squared paper. Each intersection represents a point. We aim to find a correlation between the area A of a convex shape on this grid (with vertices on the intersections), the number of grid points on the edges/ boundaries of the shape, b, and the number of grid points inside the shape, i. Based on some observations, it can be concluded that the correlation could be as follows:

$$A = i + \frac{b}{2} - 1$$

The idea of the proof is as follows: First, we construct a rectangle around our shape that completely encloses it and prove the formula for rectangles on the lattice. Then we want to show the formula for right-angled triangles. These two statements allow us to prove the formula for triangles in general. In the end we partition arbitrary convex shapes inductively into triangles and Pick's Formula is proven.

Beweis. For rectangles R we have:

$$A_R = m * n, \quad b_R = 2 * (m + n), \quad i_R = (m - 1) * (n - 1)$$

One can easily check that Pick's Formula holds for those properties:

$$\begin{aligned} i_R + \frac{b_R}{2} - 1 &= (m-1) * (n-1) + \frac{2 * (m+n)}{2} - 1 \\ &= m * n - m - n + 1 + m + n - 1 \\ &= m * n \\ &= A_R \end{aligned}$$

[1]Image from Hannah Markwig.

Now we consider a right-angled triangle T. We can construct a rectangle around it as in picture TODO. Now we have the following properties for that:

$$A_T = \frac{A_R}{2} = \frac{m*n}{2},\ b_T = \frac{b_R}{2} + a + 1 = m + n + a + 1,\ i_T = \frac{i_R - a}{2} = \frac{(m-1)*(n-1)-a}{2},$$

Where define a as the number of inner points of the rectangle which are boundary points of the triangle. For right-angled triangles we also have:

$$\begin{aligned} i_T + \frac{b_T}{2} - 1 &= \frac{i_R - a}{2} \\ &= \frac{(m-1)*(n-1)-a}{2} + \frac{m+n+a+1}{2} - 1 \\ &= \frac{m*n - m - n + 1 - a + m + n + a + 1 - 2}{2} \\ &= \frac{m*n}{2} \\ &= A_T \end{aligned}$$

For any triangle A_A, we need do distinguish between two cases (figure TODO).
For case 1 we have:

$$\begin{aligned} A_A &= A_R - A_{T_1} - A_{T_2} - A_{T_3} \\ &= m*n - \frac{m*(n-j)}{2} - \frac{j*k}{2} - \frac{n*(m-k)}{2} \\ &= \frac{2*m*n - m*n + m*j - j*k - m*n + n*k}{2} \\ &= \frac{m*j - j*k + n*k}{2} \end{aligned}$$

and

$$\begin{aligned} b_A &= b_{T_1} + b_{T_2} + b_{T_3} - b_R \\ &= m + n - j + a_1 + 1 + j + k + a_2 + 1 + n + m - k + a_3 + 1 - 2n - 2m \\ &= a_1 + a_2 + a_3 + 3 \end{aligned}$$

and

$$\begin{aligned} i_A &= i_R - a_1 - a_2 - a_3 - i_{T_1} - i_{T_2} - i_{T_3} \\ &= \frac{-a_1 - a_2 - a_3 + m*j - j*k + n*k - 1}{2} \end{aligned}$$

In the case of two points of the triangle lying on the same side of the rectangle we only have T_1 and T_2, so we leave out the terms for T_3. We check Pick's Formula:

$$\begin{aligned} i_A + \frac{b_A}{2} - 1 &= \frac{-a_1 - a_2 - a_3 + m*j - j*k + n*k - 1}{2} + \frac{a_1 + a_2 + a_3 + 3}{2} - 1 \\ &= \frac{m*j - j*k + n*k}{2} \\ &= A_A \end{aligned}$$

For case two it is similar (TODO)

□

TODO:BILD

Inhaltsverzeichnis

1 Affine geometry

In linear algebra, we studied vector spaces and linear maps. For the purpose of geometry, we sometimes also need to consider other maps, e.g. translations.

1.1 Example. Let $K \subset \mathbb{R}^2$, $K = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 - 2(x + y) + 1 = 0\}$
How can we describe K?

$$\begin{aligned} 0 &= x^2 + y^2 - 2(x + y) + 1 \\ &= x^2 - 2x + 1 + y^2 - 2y + 1 - 1 \\ &= (x - 1)^2 + (y - 1)^2 - 1 \end{aligned}$$

Set $\tilde{x} := x - 1$, $\tilde{y} := y - 1$, then

$$\tilde{K} = \{(\tilde{x}, \tilde{y}) | \tilde{x}^2 + \tilde{y}^2 - 1 = 0\}$$

is a circle around 0 with radius 1.
$\Rightarrow$ K is a circle around $(1, 1)$ with radius 1, as we can see in image 2. Our change of coordinates was just a **translation**.

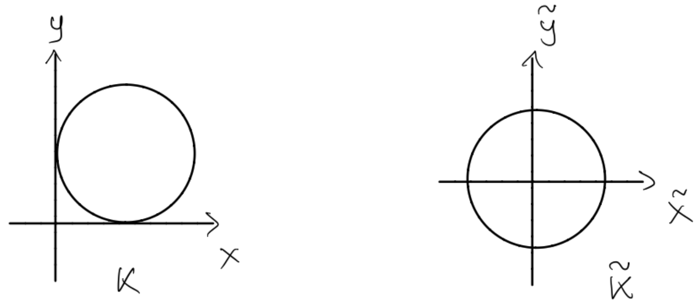


Abbildung 2: Circles K and $\tilde{K}$ with radius 1 around $(1, 1)$ and $(0, 0)$.[2]

To describe such transormations, we need **affine spaces**, roughly: linear spaces (i.e. vector spaces) without a specified 0-point. To introduce those, we need group actions:

1.2 Definition. Let $(G, *)$ be a group and M a set. A **group action** of G on M is a map

$$\begin{aligned} \cdot : G \times M &\to M : \\ (g, m) &\mapsto g \cdot m \end{aligned}$$

satisfying the following properties:

(1) $e \cdot m = m, \forall\, m \in M$

(2) $(a * b) \cdot m = a \cdot (b \cdot m), \forall\, a, b \in G,\ m \in M$

1.3 Remark. One can view a group action as a group homomorphism:

$$\begin{aligned} \varphi : G &\to \mathbb{S}(M) = \{f : M \to M, f \text{ bijective}\} \\ g &\mapsto \varphi(g) : M \to M,\ m \mapsto g \cdot m \end{aligned}$$

[2] Image from Hannah Markwig.

- *φ is well-defined*:
  $\varphi(g)$ is injective, since $\varphi(g)(m_1) = g \cdot m_1 = g \cdot m_2 = \varphi(g)(m_2)$ implies

$$\begin{aligned} g^{-1} \cdot (g \cdot m_1) &= g^{-1} \cdot (g \cdot m_2) \\ = (g^{-1} * g) \cdot m_1 &= (g^{-1} * g) \cdot m_2 \\ = e \cdot m_1 &= e \cdot m_2 \\ = m_1 &= m_2 \end{aligned}$$

  $\varphi(g)$ is surjective, since for $m \in M$ we have $g^{-1} \cdot m \in M$,

$$\begin{aligned} \varphi(g)(g^{-1} \cdot m) &= g \cdot (g^{-1} \cdot m) = (g * g^{-1}) \cdot m = e \cdot m \\ &= m \end{aligned}$$

  $\Rightarrow \varphi(g) \in \mathbb{S}(M)$.

- *φ is a group homomorphism*:

$$\begin{aligned} &\varphi(g * h) : M \to M : m \mapsto (g * h) \cdot m = g \cdot (h \cdot m) \\ &\quad = \\ &\varphi(g) \circ \varphi(h) : M \to M : m \mapsto \varphi(g)(\varphi(h)(m)) = g \cdot (h \cdot m) \end{aligned}$$

  Vice versa, a group homomorphism $\varphi : G \to \mathbb{S}(M)$ defines a group action via $g \cdot m := \varphi(g)(m)$, since

$$\begin{aligned} e \cdot m &= \varphi(e)(m) = \mathrm{id}(m) = m \\ \text{and} \quad (g * h) \cdot m &= \varphi(g * h)(m) = \varphi(g) \circ \varphi(h)(m) = g \cdot (h \cdot m) \end{aligned}$$

1.4 Example. (1) $\mathrm{GL}_n(K)$ (= invertible $n \times n$-matrices over a field K) acts in K^n via

$$A * x := A \cdot x$$

(with the group action on the left side and the matrix multiplication on the right), since $\mathbb{1}_n \cdot x = x$ and $(A \cdot B) \cdot x = A \cdot (B \cdot x)$.

(2) $\mathbb{S}_n$ (= $\mathbb{S}(\{1, \dots, n\})$) acts on $\{1, \dots, n\}$ via

$$\sigma \cdot i := \sigma(i),$$

since $\mathrm{id} \cdot i = \mathrm{id}(i) = i$ and $(\sigma \circ \sigma') \cdot i = \sigma(\sigma'(i)) = \sigma \cdot (\sigma' \cdot i)$.

(3) $\mathbb{S}_n$ acts on $\mathbb{R}^n$ via the linear map for $\sigma \in \mathbb{S}_n$ which linearly extends the permutation $e_i \mapsto e_{\sigma(i)}$ of the unit vectors. This yields a permutation matrix A_σ, i.e. in every row and column there is precisely one 1 and only 0 else.

Example: $n = 3$, $\sigma = (12)$, $A_\sigma = A_{(12)} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

For $x \in \mathbb{R}^n$ we have

$$\mathrm{id} \cdot x = A_{\mathrm{id}} \cdot x = \mathbb{1}_n \cdot x = x$$

and

$$(\sigma \circ \sigma') \cdot x = (A_\sigma \cdot A_{\sigma'}) \cdot x = A_\sigma \cdot (A_{\sigma'} \cdot x) = \sigma \cdot (\sigma' \cdot x).$$

Example: $n = 2$, $\mathbb{S}_2 = \{\mathrm{id}, (12)\}$

1.5 Definition. The **orbit** of a point $m \in M$ is

$$G \cdot m := \{g \cdot m \in M | g \in G\}$$

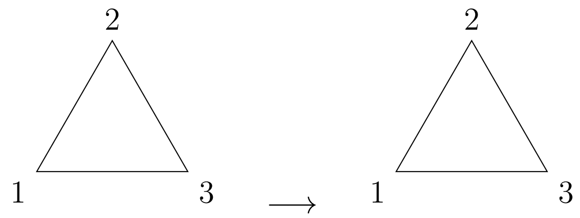
The **stabilizer** of m is

$$\mathrm{Stab}(m) := \{g \in G | g \cdot m = m\}$$

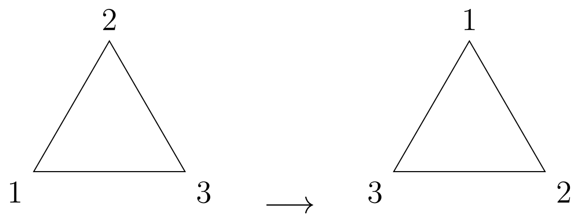
1.6 Example. The symmetry group of an equilateral triangle $\triangle \subset \mathbb{R}^2$ is

$$\mathbb{S}_3 = \{\mathrm{id}, (123), (132), (12), (13), (23)\}$$

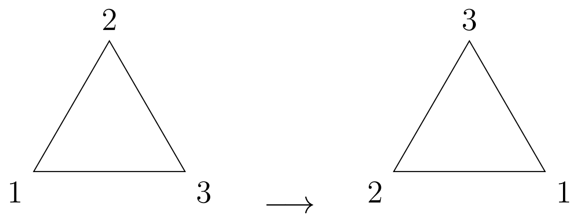
and acts on $\triangle$. It consists of rotations:

id: 

(around 0°).

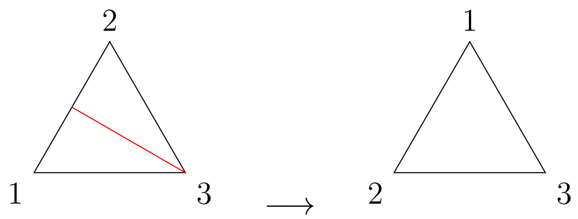
(123): 

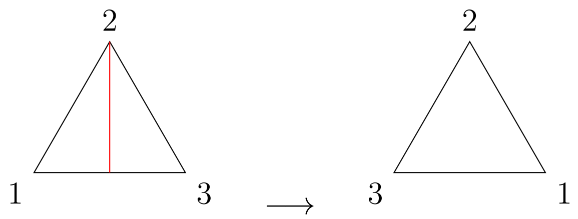
(around 120°).

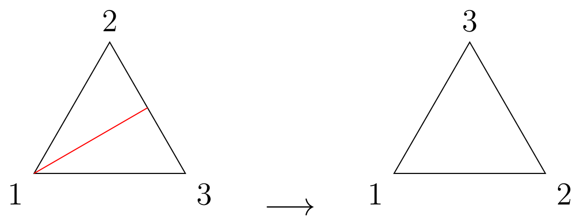
(132): 

(around 240°).

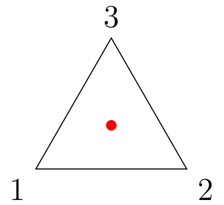
And reflections:

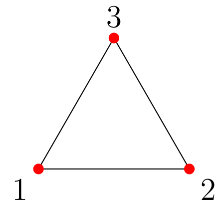
(12): 

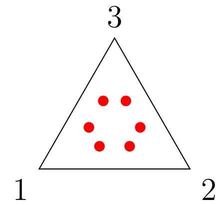
(13): 

(23): 

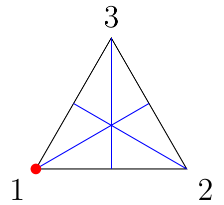
And is a group action.
We consider orbits of several points (i.e. apply all group actions and see where the point lands):

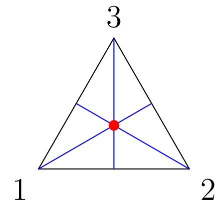


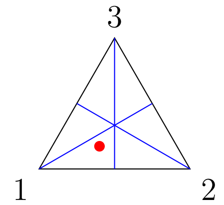




and stabilizers:







$\mathrm{Stab}(1) = \{\mathrm{id}, (23)\}$, $\quad$ $\mathrm{Stab} = \mathbb{S}_3$, $\quad$ $\mathrm{Stab} = \{\mathrm{id}\}$

1.7 Lemma. *Let $G \times M \to M$ be a group action, $m \in M$. Stab(m) is a subgroup of G.*

Beweis. $\mathrm{Stab}(m) \neq \emptyset$, since $e = \mathrm{id} \in \mathrm{Stab}(m)$. Let $a, b \in \mathrm{Stab}(m) \Rightarrow am = m$, $bm = m \Rightarrow (ab)m = a(bm) = am = m \Rightarrow ab \in \mathrm{Stab}(m)$ and
$a^{-1}m = a^{-1}(am) = (a^{-1}a)m = em = m \Rightarrow a^{-1} \in \mathrm{Stab}(m)$. □

1.8 Lemma. *Orbits are either equal or disjoint:*

$$Gm_1 = Gm_2 \quad \text{or} \quad Gm_1 \cap Gm_2 = \emptyset$$

Beweis. Assume $Gm_1 \cap Gm_2 \neq \emptyset$, then $\exists\, m_3 \in Gm_1 \cap Gm_2 \Rightarrow \exists\, g_1, g_2 \in G$ such that $m_3 = g_1 m_1 = g_2 m_2 \Rightarrow m_2 = g_2^{-1} g_1 m_1 \in Gm_1 \Rightarrow Gm_2 \subset Gm_1$
and analogously $Gm_1 \subset Gm_2$, hence equality. □

1.9 Definition. Let V be a vector space $/K$ (over a field K), A a non-empty set and

$$\tau : (V, +) \times A \to A$$

an action of the additive group of V on A. (A, V, τ) is called an **affine space**, with translation vector space V, if

$$\forall\, p, q \in A\ \exists!\, v \in V : \tau(v, p) = q$$

The conditions for the group actions are

(1) $\forall\, p \in A : \tau(0, p) = p$

(2) $\forall\, p \in A, v_1, v_2 \in V, \tau(v_1, \tau(v_2, p)) = \tau(v_1 + v_2, p)$

1.10 Example. Let V be a vector space , $A = V$ and $\tau = + : (V, +) \times V \to V$ the addition. Then $(V, V, +)$ is an affine space.
Remark: Every other example can be interpreted as this.

1.11 Remark. (1) The unique v such that $\tau(v, p) = q$ is written as $\overrightarrow{pq}$.

(2) Choose $p \in A$, define

$$F_p : V \to A : v \mapsto \tau(v, p).$$

Then F_p is bijective, since $\forall\, q\ \exists!\, v$ such that $\tau(v, p) = q$.
We can think of p as a choice of 0-point, the inverse F_p^{-1} produces V from A after the choice of a 0-point. In this sense, an affine space is like a vector space without a choice of a 0-point.

(3) One can show that every affine space is of the form $(V, V, +)$ for some vector space. We focus on affine spaces of this form and we use the notation $\mathbb{A} = \mathbb{A}(V) = (V, V, +)$

1.12 Definition. Let $\mathbb{A}(V)$ be the affine space of the vector space V. A subset $W \subset V$ is called an **affine subspace** if there exists $w \in W$ and a linear subspace $U \subset V$ s.t.

$$W = w + U = \{w + u | u \in U\}.$$

The linear subspace U is the **subspace of translations/directions** of W. The **dimension** of W is defined to be the dimension of its subspace of translations, U,
i.e. $\dim(W) = \dim(U)$.
A 0-dimensional affine subspace is called a **point**. Points are sets $x + \{0\} = \{x + 0\} = \{x\}$, $x \in V$. A 1-dimensional subspace is called **line**, a 2-dimensional **plane**. If $\dim \mathbb{A}(V) = \dim V = n$, an $(n-1)$-dimensional affine subspace is called **hyperplane**.
For $W = w + U$, w is not unique, one can pick any other element of the form $w' = w + u$ for $u \in U$ and write $W = w' + U$. The subspace of translations is unique.

1.13 Example. (1) In $\mathbb{A}(\mathbb{R}^n)$, a shift of a linear subspace is an affine space:

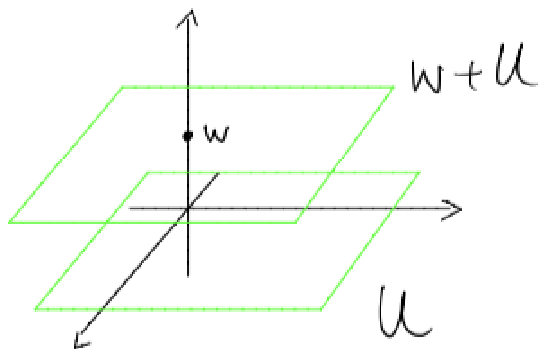


Abbildung 3: An affine space [3]

(2) Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map, $y \in \mathbb{R}^m$. Then:

$$f^{-1}(y) = \{x \in \mathbb{R}^n | f(x) = y\} = x' + \ker f,$$

for some x' with $f(x') = y$, is an affine subspace of $\mathbb{R}^n$.

[3] Image from Hannah Markwig.

(3) Solution sets of inhomogeneous systems of linear equations $Ax = b$, $b \neq 0$, are affine spaces, they are of the form $x_0 + \{x | Ax = 0\} = x_0 +$ solution set of the corresponding homogeneous system $= x_0 + \ker f$, where x_0 satisfies $Ax_0 = b$.

(4) Sometimes it is useful to consider $\mathbb{A}(\mathbb{R}^n)$ as an affine subspace of $\mathbb{A}(\mathbb{R}^{n+1})$:

$$
\begin{aligned}
\mathbb{A}(\mathbb{R}^n) &= \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_n \\ 1 \end{pmatrix} | x_i \in \mathbb{R} \right\} \\
&= \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} + \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_n \\ 0 \end{pmatrix} | x_i \in \mathbb{R} \right\} \subset \mathbb{R}^{n+1}
\end{aligned}
$$

1.14 Lemma. *Let $W_1, \ldots, W_k \subset \mathbb{A}(\mathbb{R}^n)$ be affine subspaces. Then either $W_1 \cap \ldots \cap W_k = \emptyset$ or $W_1 \cap \ldots \cap W_k$ is an affine subspace whose space of translations is $U_1 \cap \ldots \cap U_k$ (for $W_i = w_i + U_i$, U_i space of translations of W_i).*

Beweis. Let $W_1 \cap \ldots \cap W_k \neq \emptyset$. Then $\exists\, x \in W_1 \cap \ldots \cap W_k$. We can write $W_i = x + U_i$ then

$$
\begin{aligned}
W_1 \cap \ldots \cap W_k &= (x + U_1) \cap \ldots \cap (x + U_k) \\
&= x + (\bigcap_{i=1}^{k} U_i)
\end{aligned}
$$

is an affine space with space of transitions $U_1 \cap \ldots \cap U_k$. The intersection $U_1 \cap \ldots \cap U_k$ is a linear subspace, hence $W_1 \cap \ldots \cap W_k$ is an affine subspace. □

1.15 Example. (1) An empty intersection is possible , e.g. two parallel lines in $\mathbb{R}^2$,

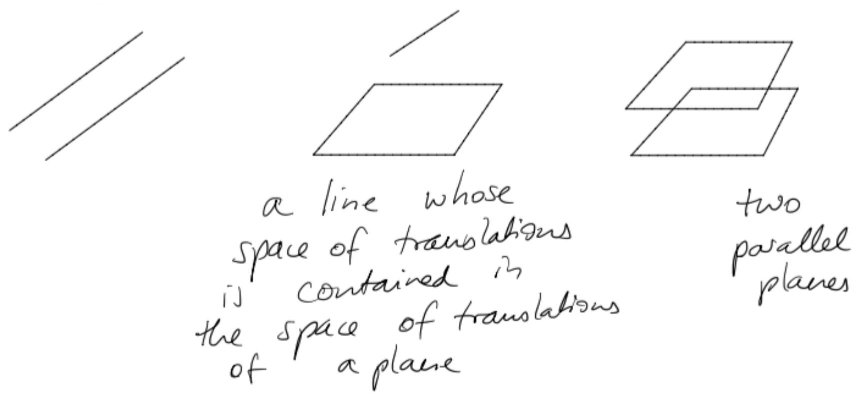


Abbildung 4: Parallel lines and planes[4]

(Parallel: the space of translations is equal (or more generally: contained in each other))

[4] Image from Hannah Markwig.

(2) In $\mathbb{A}(\mathbb{R}^4)$, let

$$W_1 = \begin{pmatrix} 2 \\ 0 \\ 0 \\ 1 \end{pmatrix} + \left\langle \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right\rangle, \quad W_2 = \begin{pmatrix} 3 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \left\langle \begin{pmatrix} -1 \\ 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\rangle,$$

We compute $W_1 \cap W_2$.
$\exists\, x \in W_1 \cap W_2 \Leftrightarrow \exists\, \alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2 \in \mathbb{R}$:

$$\begin{pmatrix} 2 \\ 0 \\ 0 \\ 1 \end{pmatrix} + \alpha_1 \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} + \alpha_3 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} = x =$$

$$\begin{pmatrix} 3 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \beta_1 \begin{pmatrix} -1 \\ 1 \\ 2 \\ 0 \end{pmatrix} + \beta_2 \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

This is an inhomogeneous system of linear equations:

$$\left(\begin{array}{ccccc|c} 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & -1 & -1 & 1 \\ 0 & 1 & 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{array}\right) \longrightarrow \left(\begin{array}{ccccc|c} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -2 & -1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right)$$

Transformed with the Gaussian elimination. The columns are the vectors after $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2$ and $(3\ 1\ 0\ 0)^T - (2\ 0\ 0\ 1)^T$. We can conclude $\beta_2 = 1$, therefore β_1 is a free parameter. With that we get $\alpha_3 = -1$, $\alpha_2 = 2\beta_1 + 1$ and $\alpha_1 = -\beta_1 + 1$.

$$\Rightarrow \quad W_1 \cap W_2 = \left\{ x = \begin{pmatrix} 3 \\ 2 \\ 0 \\ 0 \end{pmatrix} + \beta_1 \begin{pmatrix} -1 \\ 1 \\ 2 \\ 0 \end{pmatrix} | \beta_1 \in \mathbb{R} \right\}$$

1.16 Lemma. *Let $\mathbb{A} = \mathbb{A}(V) = (V, V, +) = \mathbb{K}^n$ be the affine space of the vector space V. $W \subset \mathbb{A}(V)$ is an affine subspace.*

$$\Leftrightarrow \quad \forall\, k \geq 1,\, a_1, \ldots, a_k \in W,\, \lambda_1, \ldots, \lambda_k \in \mathbb{K} : \sum_{i=1}^{k} \lambda_i = 1 \Rightarrow \sum_{i=1}^{k} \lambda_i a_i \in W$$

Beweis. " $\Rightarrow$ " Let $W = w + U$ be an affine subspace. Let $k \geq 1$,
$a_1, \ldots, a_k \in W$, $\lambda_1, \ldots, \lambda_k \in \mathbb{K}$ with $\sum_{i=1}^{k} \lambda_i = 1$. Then $a_i = w + u_i$ for $u_i \in U$, and

$$\sum_{i=1}^{k} \lambda_i a_i = \sum_{i=1}^{k} \lambda_i (w + u_i) = \underbrace{(\sum_{i=1}^{k} \lambda_i) w}_{w} + \underbrace{\sum_{i=1}^{k} \lambda_i u_i}_{\in U} = w + \sum_{i=1}^{k} \lambda_i u_i = w + U = W$$

" $\Leftarrow$ " Let $w \in W$, set $U := W - w = \{w' - w | w' \in W\}$. Then $W = w + U$.
We show: U is a linear subspace of V. Let $w_1 - w, w_2 - w \in U$, $\lambda \in K$. Then

$$\lambda(w_1 - w) \in U, \text{ since } w + (\lambda(w_1 - w)) = (1 - \lambda) w + \lambda w_1,\ w, w_1 \in W$$

and by our condition, $(1-\lambda)w + \lambda w_1 \in W$. Also,

$$w_1 - w + w_2 - w \in U \text{ since } w + w_1 - w + w_2 - w = w_1 + w_2 - w,$$

but $w_1, w_2, w \in W$ and since $1+1-1 = 1$ also by our condition $w_1 + w_2 - w \in W$. $\square$

1.17 Remark. (1) We call $\sum_{i=1}^k \lambda_i a_i$ with $\sum_{i=1}^k \lambda_i = 1$ an **affine combination** of the a_i. Lemma 1.16 then reads:
W is an affine subspace $\Leftrightarrow$ W is closed under taking affine combinations.

(2) Lemma 1.16 gives insights why it is useful to identify $\mathbb{A}(\mathbb{R}^n)$ with the affine hyperplane $e_{n+1} + \langle e_1, \ldots, e_n \rangle \subset \mathbb{R}^{n+1}$:

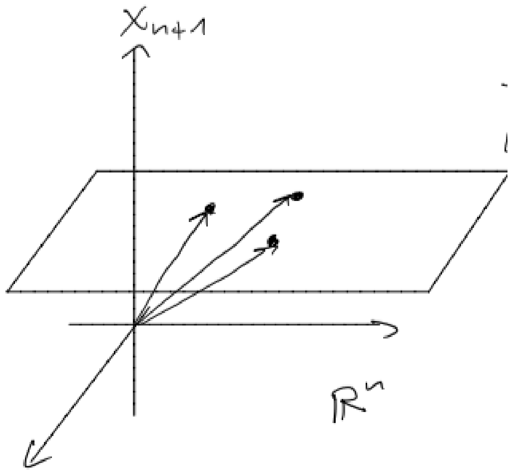


Abbildung 5: With plane $H = \{x_{n+1} = 1\}$ [5]

Points p_i in $\mathbb{A}(\mathbb{R}^n)$ come from points $\begin{pmatrix} p_i \\ 1 \end{pmatrix} \in \mathbb{R}^{n+1}$. A linear combination of such points, $\sum \lambda_i \begin{pmatrix} p_i \\ 1 \end{pmatrix} = \begin{pmatrix} \sum \lambda_i p_i \\ \sum \lambda_i \end{pmatrix}$ is in $H = \mathbb{A}(\mathbb{R}^n) \Leftrightarrow \sum \lambda_i = 1$. For this model of $\mathbb{A}(\mathbb{R}^n)$, we can understand affine combinations as linear combinations which stay within H.

1.18 Definition. Let $\mathbb{A} = \mathbb{A}(V)$ be an affine space, $\emptyset \neq X \subset V$ subset, the **affine hull** $\operatorname{Aff}(X)$ is the smallest affine subspace of $\mathbb{A}$ containing X, i.e.

$$\operatorname{Aff}(X) = \bigcap_{\substack{W \subset \mathbb{A} \text{ affine subspace} \\ X \subset W}} W$$

1.19 Lemma.

$$\operatorname{Aff}(X) = \{\sum \lambda_i p_i | p_i \in X, \sum \lambda_i = 1\}$$

(Affine combinations are finite by definition, therefore the sum is finite.)

[5] Image from Hannah Markwig.

Beweis. $\{\sum \lambda_i p_i | p_i \in X, \sum \lambda_i = 1\}$:=RHS (right hand side) satisfies the condition from lemma 1.16 (it is closed under taking affine combinations, $\Rightarrow$ RHS is an affine subspace, $X \subset$ RHS) and is thus an affine space, also it contains X. Hence $\mathrm{Aff}(X) \subset$ RHS.
Furthermore, by lemma 1.16, every affine subspace W containing X also contains RHS

$$\Rightarrow \{\sum \lambda_i p_i | p_i \in X, \sum \lambda_i = 1\} \subset \bigcap_{\substack{W \text{ affine subspace} \\ X \subset W}} = \mathrm{Aff}(X)$$

$\square$

1.20 Definition. Let $\mathbb{A} = \mathbb{A}(V)$ be an affine space. $p_0, \ldots, p_r$ are **affinely independent**, iff ($\Leftrightarrow$)

$$\sum_{i=0}^{r} \lambda_i p_i = 0 \quad \text{and} \quad \sum_{i=0}^{r} \lambda_i = 0 \Rightarrow \lambda_0 = \ldots = \lambda_r = 0$$

1.21 Lemma. *$\{p_0, \ldots, p_r\}$ are affinely independent $\Leftrightarrow$ $\{p_1 - p_0, \ldots, p_r - p_0\}$ are linearly independent.*

Beweis. " $\Rightarrow$ " Let $\{p_0, \ldots, p_r\}$ be affinely independent. Consider

$$\begin{aligned} &\lambda_1(p_1 - p_0) + \ldots + \lambda_r(p_r - p_0) = 0 \\ = &\underbrace{(-\lambda_1 - \ldots - \lambda_r)}_{=:\lambda_0} p_0 + \lambda_1 p_1 + \ldots + \lambda_r p_r = 0 \end{aligned}$$

Then

$$\sum_{i=0}^{r} \lambda_i p_i = 0 \quad \text{and} \quad \sum_{i=0}^{r} \lambda_i = -\lambda_1 - \ldots - \lambda_r + \lambda_1 + \ldots + \lambda_r = 0$$

so $\lambda_0 = \ldots = \lambda_r = 0$ and $\{p_1 - p_0, \ldots, p_r - p_0\}$ are linearly independent.
" $\Leftarrow$ " Let $\{p_1 - p_0, \ldots, p_r - p_0\}$ be linearly independent. Let $\sum_{i=0}^{r} \lambda_i p_i = 0$ and $\sum_{i=1}^{r} \lambda_i = 0$. We can write $\lambda_0 = -\lambda_1 - \ldots - \lambda_r$, so

$$\begin{aligned} 0 = \sum_{i=0}^{r} \lambda_i p_i &= \lambda_0 p_0 + \lambda_1 p_1 + \ldots + \lambda_r p_r \\ &= (-\lambda_1 - \ldots - \lambda_r) p_0 + \lambda_1 p_1 + \ldots + \lambda_r p_r \\ &= \lambda_1(p_1 - p_0) + \ldots + \lambda_r(p_r - p_0) \end{aligned}$$

$\Rightarrow \ \lambda_1 = \ldots = \lambda_r = 0 \ \Rightarrow \ \lambda_0 = 0$ $\square$

1.22 Definition. Let $\mathbb{A} = \mathbb{A}(V)$ be an affine space, $W \subset \mathbb{A}$ an affine subspace of dimension k, $p_0, \ldots, p_k \in W$ affinely independent. Then $p_0, , \ldots, p_k$ is an **affine basis** of W.
Every point $w \in W$ can uniquely be written as $w = p_0 + \sum \mu_i (p_i - p_0)$, $\mu_i \in K$. $(\mu_1, \ldots \mu_n)$ are the **affine coordinates** of w w.r.t. the basis $p_0, \ldots, p_k$.

1.23 Example. A chair with 3 legs never wobbels: 3 distinct points p_0, p_1, p_2 which do not span a line are affinely independent and thus span an affine plane. (see exercises)

1.24 Remark. We can also define affine coordinates of $W \subset \mathbb{A}$ by using $W = p_0 + U$, and a basis $p_1, \ldots, p_k$ of the linear subspace U. Then $p_i := p_i' + p_0 \in W$. We now consider maps that preserve the structure. For example, affine hulls should be mapped to affine hulls.

1.25 Definition. Let V, W be K-vectorspaces, $\mathbb{A}(V), \mathbb{A}(W)$ the corresponding affine spaces. $f : \mathbb{A}(V) \to \mathbb{A}(W)$ is an **affine map**, if $\forall\, x, y \in \mathbb{A}(V)$, $\lambda, \mu \in K$:

$$\lambda + \mu = 1 : \quad f(\lambda x + \mu y) = \lambda f(x) + \mu f(y)$$

Inductively we have

$$f(\sum \lambda_i p_i) = \sum \lambda_i f(p_i) \quad \text{for} \quad \sum \lambda_i = 1.$$

1.26 Example. Let $\mathbb{A}(V)$ be an affine space, $b \in V$:

(1) The translation $T_b : \mathbb{A}(V) \to \mathbb{A}(V) : x \mapsto x + b$ is affine, since for $\lambda + \mu = 1$ we have

$$\begin{aligned} T_b(\lambda x + \mu x) =& \lambda x + \mu y + b \\ =& \lambda x + \lambda b + \mu y + (1 - \lambda) b \\ =& \lambda x + \lambda b + \mu y + \mu b \\ =& \lambda(x + b) + \mu(y + b) \\ =& \lambda T_b(x) + \mu T_b(y) \end{aligned}$$

(2) If we stretch with center z and factor λ, i.e. $z \mapsto z$ and for $x \neq z$ we map x to the point y on the line through x and z for which $y - z = \lambda(x - z)$, we obtain an affine map

$$\begin{aligned} &\sigma : \mathbb{A}(V) \to \mathbb{A}(V) : x \mapsto \lambda x + (1 - \lambda) z : \\ &\sigma(\mu x_1 + (1 - \mu) x_2) = \lambda(\mu x_1 + (1 - \mu) x_2) + (1 - \lambda) z \\ &\qquad = \mu(\lambda x_1 + (1 - \lambda) z) + (1 - \mu)(\lambda x_2 + (1 - \lambda) z) \\ &\qquad = \mu \sigma(x_1) + (1 - \mu) \sigma(x_2) \end{aligned}$$

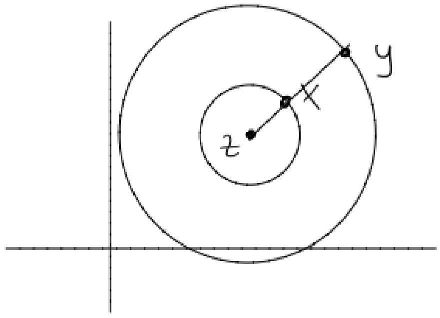


Abbildung 6: Map x on y [6]

If $\lambda = 0$ everything is contracted to the point z, if $\lambda = 1$, $\sigma = \mathrm{id}$. If $\lambda = -1$ this is the reflection with center z.

1.27 Theorem. *$\mathbb{A} = \mathbb{A}(V)$, choose an origin $p_0 \in \mathbb{A}$. $f : \mathbb{A}(V) \to \mathbb{A}(W)$ is affine $\Leftrightarrow$ $\exists!$ linear map $\phi : V \to W$ s.t. $f(p_0 + x) = \phi(x) + f(p_0)$. ϕ does not depend on the choice of p_0. It is called the* ***linear part*** *of f.*

[6] Image from Hannah Markwig.

Beweis. " $\Leftarrow$ " Let $p = p_0 + x$, $q = p_0 + y$,

$$\begin{aligned}
f(\lambda p + (1-\lambda)q) =& f(\lambda p_0 + \lambda x + (1-\lambda)p_0 + (1-\lambda)y) \\
=& f(p_0 + \lambda x + (1-\lambda)y) \\
=& \phi(\lambda x + (1-\lambda)y) + f(p_0) \\
=& \lambda\phi(x) + (1-\lambda)\phi(y) + \lambda f(p_0) + (1-\lambda)f(p_0) \\
=& \lambda(\phi(x) + f(p_0)) + (1-\lambda)(\phi(y) + f(p_0)) \\
=& \lambda f(p) + (1-\lambda)f(q)
\end{aligned}$$

hence f is affine.
" $\Rightarrow$ " Define

$$\phi : V \to W : \ x \mapsto f(p_0 + x) - f(p_0)$$

We show ϕ is linear, i.e.

$$\forall\, x, y \in V,\ \phi(x+y) = \phi(x) + \phi(y),\ \phi(\lambda x) = \lambda\phi(x),\ \lambda \in K :$$

We have $p_0, p_0 + x, p_0 + y \in \mathbb{A}(V)$. Thus (with f affine and $1+1-1=1$ in the fourth line)

$$\begin{aligned}
\phi(x+y) =& f(p_0 + x + y) - f(p_0) \\
=& f(p_0 + x + p_0 + y - p_0) - f(p_0) \\
=& f(1 * (p_0 + x) + 1 * (p_0 + y) - 1 * p_0) - f(p_0) \\
=& f(p_0 + x) + f(p_0 + y) - f(p_0) - f(p_0) \\
=& f(p_0 + x) - f(p_0) + f(p_0 + y) - f(p_0) \\
=& \phi(x) + \phi(y)
\end{aligned}$$

for the additivity. And for the linearity:

$$\begin{aligned}
\phi(\lambda x) =& f(p_0 + \lambda x) - f(p_0) \\
=& f(\lambda(p_0 + x) + (1-\lambda)p_0) - f(p_0) \\
=& \lambda f(p_0 + x) + (1-\lambda)f(p_0) - f(p_0) \\
=& \lambda(f(p_0 + x) - f(p_0)) \\
=& \lambda\phi(x)
\end{aligned}$$

Uniqueness: Assume $p_1 \in V$ is a different choice of origin. We obtain a linear map

$$\Psi : V \to W : \ x \mapsto f(p_1 + x) - f(p_1).$$

Then

$$\begin{aligned}
\Psi(x) =& f(p_1 + x) - f(p_1) \\
=& f(p_0 + (p_1 - p_0) + x) - f(p_1) \\
=& \phi((p_1 - p_0) + x) + f(p_0) - f(p_1) \\
=& \phi(x) + \phi(p_1 - p_0) + f(p_0) - f(p_1) \\
=& \phi(x) + f(p_1) - f(p_0) + f(p_0) - f(p_1) \\
=& \phi(x)
\end{aligned}$$

□

1.28 Remark. By theorem 1.27, the choice of origin is irrelevant in the context of affine maps. We can thus choose 0 as origin and obtain

$$f(x) = \phi(x) + f(0) = T_b \circ \phi(x), \quad b := f(0),$$

thus every affine map is a linear map followed by a translation. In particular, affine maps $\mathbb{A}(K^n) \to \mathbb{A}(K^m)$ are of the form $x \mapsto Ax + b$, where $A \in \mathrm{Mat}(m \times n, K), b \in K^m$.
We now consider affine maps $\mathbb{A}(V) \to \mathbb{A}(V)$ which are bijective:
Let $B = \{p_1, \ldots, p_n\}$ be a basis of V, p_0 a choice of origin. Every $x \in \mathbb{A}(V)$ can uniquely be written as

$$X = p_0 + \sum_{j=1}^{n} \mu_j p_j.$$

In affine coordinates, x is thus given as $\begin{pmatrix} \mu_1 \\ \vdots \\ \mu_n \end{pmatrix} \in K^n$.
Let $A = {}_B M_B(\phi)$, where ϕ denotes the linear part of the affine map f.
Let $f(p_0) = p_0 + \sum_{i=1}^{n} b_i p_i$. Then

$$\begin{aligned} f(x) =& f(p_0 + \sum_{i=1}^{n} \mu_i p_i) \\ =& f(p_0) + \phi(\sum \mu_i p_i) \\ =& p_0 + \sum_{i=1}^{n} b_i p_i + \sum \mu_i \phi(p_i) \\ =& p_0 + \sum_{i=1}^{n} b_i p_i + \sum_{j=1}^{n} \mu_j (\sum_{i=1}^{n} a_{ij} p_i) \\ =& p_0 + \sum_{i=1}^{n} (\sum_{j=1}^{n} a_{ij} \mu_j + b_i) p_i \end{aligned}$$

The affine coordinates of $f(x)$ are thus $A * \mu + b$, where μ are the affine coordinates of x.

1.29 Remark. (1) Translations T_b are bijective.

(2) f is bijective $\Leftrightarrow$ the linear part ϕ is bijective.

1.30 Theorem. *The bijective affine maps on $\mathbb{A}(V)$ form a group.*

Beweis. Choose 0 as the origin of $\mathbb{A}(V)$. Then every bijective affine map is of the form $T_b \circ \phi$ for a bijective linear part ϕ.
$id_V : \mathbb{A}(V) \to \mathbb{A}(V)$ is a bijective affine map. For $f = T_{b_1} \circ \phi_1$, $g = T_{b_2} \circ \phi_2$ we have

$$f \circ g = T_{b_1} \circ \phi_1 \circ T_{b_2} \circ \phi_2 = T_{b_1} \circ (\phi_1 \circ T_{b_2}) \circ \phi_2 = T_{b_1} \circ T_{\phi_1(b_2)} \circ \phi_1 \circ \phi_2$$

since

$$\phi_1 \circ T_{b_2}(X) = \phi_1(X + b_2) = \phi_1(X) + \phi_1(b_2) = T_{\phi_1(b_2)} \circ \phi_1(X)$$

Furthermore,

$$T_{b_1} \circ T_{\phi_1(b_2)} \circ \phi_1 \circ \phi_2 = T_{b_1+\phi_1(b_2)} \circ (\phi_1 \circ \phi_2)$$

is again affine and bijective. For $f = T_b \circ \phi$ we have

$$f^{-1} = (T_b \circ \phi)^{-1} = \phi^{-1} \circ T_b^{-1} = \phi^{-1} \circ T_{-b} = T_{\phi^{-1}(-b)} \circ \phi^{-1}$$

is again affine and bijective.
The set of bijective affine maps, viewed as a subset of the group of bijective maps on V is thus nonempty, closed under composition and closed under taking inverses. Hence it is a subgroup. $\square$

1.31 Example. A bijective affine map f of $\mathbb{A}(K^n)$ is of the form $x \mapsto Ax + b$, $A \in \mathrm{Mat}(n \times n, K)$. We write $\mathbb{A}(K^n)$ as an affine subspace of K^{n+1}:

$$\mathbb{A}(K^n) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} + \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_n \\ 0 \end{pmatrix} | x_i \in K \right\}$$

and define

$$\tilde{A} := \begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix} \in \mathrm{Mat}((n+1) \times (n+1), K).$$

Then

$$\tilde{A} \begin{pmatrix} x \\ 1 \end{pmatrix} = \begin{pmatrix} Ax + b \\ 1 \end{pmatrix},$$

thus $\tilde{A}$ maps $\mathbb{A}(K^n)$ into itself and we can combine the linear part and the translation part of f into one matrix.

1.32 Definition. Two affine subspaces $W_1 = w_1 + U_1$, $W_2 = w_2 + U_2$ are **parallel**, $W_1 || W_2$, if $U_1 \subset U_2$ or $U_2 \subset U_1$.

1.33 Remark. Being parallel is reflexive and by definition symmetric, but not necessarily transitive, e.g.

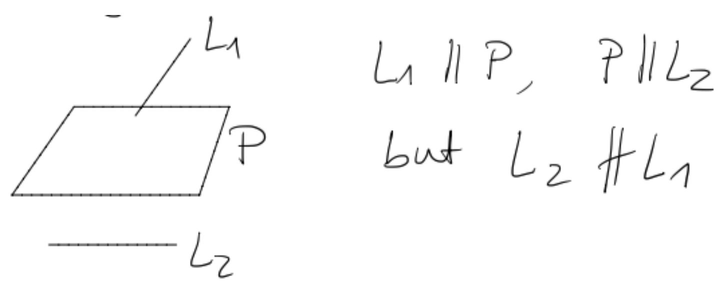


Abbildung 7: Lines parallel to a plane [7]

If we restrict the relation to subspaces of the same dimension, then it is transitive. Then being parallel is an equivalence relation.

[7]Image from Hannah Markwig.

1.34 Definition. Let p, q, r be collinear (i.e. on a line) and $p \neq q$. If $q = p + x$ and $r = p + \lambda x$, then λ is called the **ratio** of r and q w.r.t. p.

1.35 Lemma. *(1) Affine maps map parallel affine subspaces to parallel affine subspaces.*

(2) Bijective affine maps preserve ratios.

Beweis. (1) Let $f : \mathbb{A}(V) \to \mathbb{A}(V)$ ba an affine map, ϕ its linear part. Let $W_1 = w_1 + U_1$ be an affine subspace, then $f(W_1) = f(w_1) + \phi(U_1)$. For $W_1 || W_2$, $W_2 = w_2 + U_2$, we have $U_1 \subset U_2$ without restriction

$$\Rightarrow \phi(U_1) \subset \phi(U_2) \Rightarrow f(W_1) || f(W_2).$$

(2) Let p, $q = p + x$, $r = p + \lambda x$ be on a line. By (1), $f(p)$, $f(q)$ and $f(r)$ are on a line, and $f(p) \neq f(q)$. Furthermore,

$$f(q) = f(p) + \phi(x),\ f(r) = f(p) + \phi(\lambda x) = f(p) + \lambda \phi(x)$$

$\Rightarrow \lambda$ is also the ratio of $f(q)$ and $f(r)$ w.r.t. $f(p)$.

□

Now we restrict to $K = \mathbb{R}$ and consider Euclidean vector spaces $(V, \langle \cdot, \cdot \rangle)$, V with a scalar product $\langle \cdot, \cdot \rangle$. V is then also a metric space with

$$\mathrm{d}(x, y) = \|x - y\| = \sqrt{\langle x - y, x - y \rangle}$$

1.36 Definition. A **(Euklidean) isometry** on V (resp. $\mathbb{A}(V)$) is a bijective map f such that

$$\mathrm{d}(f(x), f(y)) = \mathrm{d}(x, y), \quad \forall\, x, y \in V.$$

1.37 Remark. Euklidean linear maps $\phi : V \to V$ satisfy

$$\langle \phi(x), \phi(y) \rangle = \langle x, y \rangle \quad \forall\, x, y \in V.$$

These are spacial cases of Euklidean isometries.

1.38 Theorem. *The Euclidean isometries form a group.*

Beweis. id is an Euklidean isometry. If f and g are, then

$$\begin{aligned} \mathrm{d}(g \circ f(x), g \circ f(y)) &= \mathrm{d}(g(f(x)), g(f(y)) \\ &\overset{\text{g isometry}}{=} \mathrm{d}(f(x), f(y)) \\ &\overset{\text{f isometry}}{=} d(x, y) \end{aligned}$$

so $g \circ f$ is an isometry. Since f is bijective, f^{-1} exixts and for $x = f(u)$, $y = f(v) \Rightarrow f^{-1}(x) = u$, $f^{-1}(y) = v$ and

$$\mathrm{d}(f^{-1}(x), f^{-1}(y)) = \mathrm{d}(f^{-1}(f(u)), f^{-1}(f(v))) = \mathrm{d}(u, v) \overset{\text{f isometry}}{=} \mathrm{d}(f(u), f(v)) = \mathrm{d}(x, y)$$

therefore f^{-1} is an isometry.
Isometries thus form a nonempty subset of the set of bijective maps which is closed under composition and taking inverses, hence a subgroup. □

1.39 Lemma. *Let $a_0, \dots, a_n$ be affinely independent in $\mathbb{A}(V)$, $\dim(V) = n$. A Euklidean isometry $f : \mathbb{A}(V) \to \mathbb{A}(V)$ is uniquely determined by the images $f(a_0), \dots, f(a_n)$.*

Beweis. Let f, g be isometries with $f(a_i) = g(a_i)$, $\forall\, i$. Then $g^{-1} \circ f(a_i) = a_i \,\forall\, i$. Let $b_i := T_{-a_0}(a_i) = a_i - a_0$, then $\{b_1, \dots, b_n\}$ is a basis of V.
<u>Claim:</u> $h := T_{-a_0} \circ g^{-1} \circ f \circ T_{a_0} = \mathrm{id}$.
Since $T_{-a_0} = (T_{a_0})^{-1}$, we then have $g^{-1} \circ f = T_{a_0} \circ T_{a_0}^{-1} = \mathrm{id}$, $\Rightarrow f = g$ proving the theorem.
We have

$$h(0) = T_{-a_0} \circ g^{-1} \circ f \circ T_{a_0}(0) = T_{-a_0} \circ (g^{-1} \circ f)(a_0) = T_{-a_0}(a_0) = 0,$$

and

$$\begin{aligned} h(b_i) &= T_{-a_0} \circ g^{-1} \circ f \circ T_{a_0}(b_i) = T_{-a_0} \circ g^{-1} \circ f(b_i + a_0) \\ &= T_{-a_0} \circ g^{-1} \circ f(a_i) = T_{-a_0}(a_i) = b_i \end{aligned}$$

Since h is an isometry, we have for $x \in V$ and $y = h(x)$, $\mathrm{d}(x, 0) = \mathrm{d}(y, 0)$, $\mathrm{d}(b_i, x) = \mathrm{d}(b_i, y)$ $\Leftrightarrow$

$$\begin{aligned} \langle x, x \rangle &= \langle y, y \rangle, \\ \langle x - b_i, x - b_i \rangle &= \langle y - b_i, y - b_i \rangle \\ &= \langle x, x \rangle - 2\langle x, b_i \rangle + \langle b_i, b_i \rangle \\ &= \langle y, y \rangle - 2\langle y, b_i \rangle + \langle b_i, b_i \rangle \\ &\Rightarrow \langle x, b_i \rangle = \langle y, b_i \rangle \,\forall\, i \end{aligned}$$

Since $\{b_1, \dots, b_n\}$ is a basis of V,

$$\langle x, z \rangle = \langle y, z \rangle \,\forall\, z \Rightarrow \langle x - y, z \rangle = 0 \,\forall\, z \in V \Rightarrow x - y = 0 \Rightarrow x = y = h(x) \,\forall\, x \in V$$

and $h = \mathrm{id}$ as claimed. $\square$

1.40 Theorem. *Every isometry is of the form $f = T_b \circ \phi$, where ϕ is an Euklidean linear map.*

Beweis. Let $\{a_1, \dots, a_n\}$ be a basis of V and $a_0 := 0$ a choice of origin. $\{a_0, \dots, a_n\}$ then form an affine basis. Let $b_0 := f(a_0) = f(0)$ and $g := T_{-b_0} \circ f$. Then $g(0) = 0$.
Furthermore, we set $b_i := f(a_i)$ for $1 \leq i \leq n$. We construct a Euclidean linear map ϕ satisfying $\phi(a_i) = g(a_i) = T_{-b_0} \circ f(a_i) \,\forall\, 0 \leq i \leq n$, by lemma 1.39 we then have $\phi = g = T_{-b_0} \circ f \Rightarrow T_{b_0} \circ \phi = f$.
Since g is an isometry, we have

$$\mathrm{d}(a_i, a_j) = \mathrm{d}(g(a_i), g(a_j)) = \mathrm{d}(f(a_i) - b_0, f(a_j) - b_0) = \mathrm{d}(b_i - b_0, b_j - b_0)$$

Let $\tilde{b_i} := b_i - b_0$, then $\mathrm{d}(a_i, a_j) = \mathrm{d}(g(a_i), g(a_j)) = \mathrm{d}(\tilde{b_i}, \tilde{b_j})$. Furthermore,

$$\langle a_i, a_i \rangle = \mathrm{d}(a_i, 0)^2 \overset{\text{f isometry}}{=} \mathrm{d}(f(a_i), f(0))^2 = \mathrm{d}(b_i, b_0)^2 = \mathrm{d}(b_i - b_0, 0)^2 = \mathrm{d}(\tilde{b_i}, 0)^2 = \langle \tilde{b_i}, \tilde{b_i} \rangle$$

and with

$$\mathrm{d}(a_i, a_j)^2 = \langle a_i - a_j, a_i - a_j \rangle = \langle a_i, a_i \rangle - 2\langle a_i, a_j \rangle + \langle a_j, a_j \rangle$$

and

$$\mathrm{d}(\tilde{b_i}, \tilde{b_j})^2 = \langle \tilde{b_i}, \tilde{b_i} \rangle - 2\langle \tilde{b_i}, \tilde{b_j} \rangle + \langle \tilde{b_j}, \tilde{b_j} \rangle$$

we can conclude

$$\langle a_i, a_j \rangle = \langle \tilde{b_i}, \tilde{b_j} \rangle \quad \forall\, i, j.$$

Let ϕ be the unique linear map satisfying $\phi(a_i) = b_i - b_0 = \tilde{b_i}\, \forall\, i = 1, \ldots, n$.
Let $x, y \in V$, write

$$x - y = \sum_{i=1}^{n} \lambda_i a_i$$

Then it follows

$$\phi(x) - \phi(y) = \phi(x - y) = \sum_{i=1}^{n} \lambda_i \tilde{b_i}$$

and therefore

$$\begin{aligned}
\mathrm{d}(\phi(x), \phi(y))^2 =& \langle \phi(x) - \phi(y), \phi(x) - \phi(y) \rangle \\
=& \langle \sum_{i=1}^{n} \lambda_i \tilde{b_i}, \sum_{j=1}^{n} \lambda_j \tilde{b_j} \rangle = \sum_{i,j=1}^{n} \lambda_i \lambda_j \langle \tilde{b_i}, \tilde{b_j} \rangle \\
=& \sum_{i,j=1}^{n} \lambda_i \lambda_j \langle a_i, a_j \rangle = \langle \sum_{i=1}^{n} \lambda_i a_i, \sum_{j=1}^{n} \lambda_j a_j \rangle \\
=& \langle x - y, x - y \rangle = \mathrm{d}(x, y)^2
\end{aligned}$$

so ϕ is Euclidean. □

To sum up we have:

Isometries $\subset$	Affine bijective maps
$f = T_b \circ \phi$ with ϕ Euclidean linear map	$f = T_b \circ \phi$ with ϕ bijective linear map
in coordinates: $x \mapsto Ax + b$, $A \in \mathrm{O}(n)$ (orthogonal)	in coordinates: $x \mapsto Ax + b$, $A \in \mathrm{GL}_n(K)$

2 Dual vector spaces

2.1 Definition. Let V be a vector space over K.

$$V^\vee := \mathrm{Hom}_K(V, K)$$

is the **dual vector space of** V. Its elements are called **linear forms**.
From linear algebra 1: $\mathrm{Hom}_K(V, W)$ is a vector space, in particular $V^\vee$ is a vector space.

2.2 Example. (1) $V = \mathbb{R}^3$, $f : \mathbb{R}^3 \to \mathbb{R} : \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto x + y + z$ is a linear form.

(2) $V = K[x]_{\leq d}$, $a \in K$, $\phi : V \to K : f \mapsto f(a)$ is a linear form.

(3) Let $V = \{f : \mathbb{R} \to \mathbb{R},\ f \text{ continuous}\}$, $\int_a^b : V \to \mathbb{R} : f \mapsto \int_a^b f(x)dx$ is a linear form.

2.3 Lemma. *Let V be finite dimensional. Then $V^\vee \cong V$.*

Beweis. $V^\vee = \mathrm{Hom}(V, K) \cong \mathrm{Mat}(1 \times n, K)$ if $\dim V = n$, $\mathrm{Mat}(1 \times n, K) \cong K^n \cong V$. $\square$

2.4 Remark. It is not its vector space structure that makes $V^\vee$ interesting, but its relation to V! We have a pairing:

$$V^\vee \times V \to K : (f, v) \mapsto f(v)$$

2.5 Definition (Dual Basis). Let $\dim_K V < \infty$, $B = \{v_1, \ldots, v_n\}$ be a basis of V. We set $v_i^\vee \in V^\vee$,

$$v_i^\vee(v_j) := \delta_{ij} := \begin{cases} 1 & i = j \\ 0 & \text{else} \end{cases}$$

the **Kronecker symbol** and $B^\vee = \{v_1^\vee, \ldots, v_n^\vee\}$.

2.6 Remark. $v_i^\vee$ depends on all of B, not just v_i. $v_i^\vee$ is defined by linear extension:

$$v_i^\vee(\sum_{j=1}^n \lambda_j v_j) = \sum_{j=1}^n \lambda_j v_i^\vee(v_j) = \sum_{j=1}^n \lambda_j \delta_{ij} = \lambda_i.$$

2.7 Lemma. *Let $\dim V < \infty$, B a basis. $B^\vee$ is a basis.*

Beweis. We show:

(1) $B^\vee$ is linearly independent.

(2) $B^\vee$ generates $V^\vee$.

Then the statement follows.

(1) Let

$$\mu_1 v_1^\vee + \ldots + \mu_n v_n^\vee = 0 \quad \Rightarrow \quad \forall\, v \in V : \mu_1 v_1^\vee + \ldots + \mu_n v_n^\vee(v) = 0$$

in particular, $\forall\, i = 1, \ldots, n$:

$$\begin{aligned} 0 &= \mu_1 v_1^\vee + \ldots + \mu_n v_n^\vee(v_i) \\ &= \mu_1 v_1^\vee(v_i) + \ldots + \mu_n v_n^\vee(v_i) \\ &= \mu_1 \delta_{1i} + \ldots + \mu_n \delta_{ni} \\ &= \mu_i \end{aligned}$$

(2) Let $f \in V^\vee$. f is determined by its values on B. Let $\mu_i = f(v_i)$, $v \in V$, $\lambda_i \in K$: $v = \sum_{i=1}^n \lambda_i v_i$. Then

$$\begin{aligned} f(v) = f(\sum \lambda_i v_i) = \sum \lambda_i f(v_i) &= \sum \lambda_i \mu_i \\ &= \sum \lambda_i (\mu_1 v_i^\vee + \ldots + \mu_n v_n^\vee(v_i)) \\ &= \mu_1 v_1^\vee + \ldots + \mu_n v_n^\vee(\sum \lambda_i v_i) \\ &= \mu_1 v_1^\vee + \ldots + \mu_n v_n^\vee(v) \end{aligned}$$

$$\Rightarrow\ f = \mu_1 v_1^\vee + \ldots + \mu_n v_n^\vee \quad \Rightarrow\ f \in \langle v_1^\vee, \ldots, v_n^\vee \rangle.$$

□

2.8 Remark. If we write $v = \sum \lambda_i v_i$ and $f = \sum \mu_j v_j^\vee$ then

$$f(v) = \sum_j \mu_j v_j^\vee(\sum_i \lambda_i v_i) = \sum_{i,j} \mu_j \lambda_i v_j^\vee(v_i) = \sum_{i,j} \mu_j \lambda_i \delta_{ij} = \sum_i \mu_i \lambda_i = \mu_1 \lambda_1 + \ldots + \mu_n \lambda_n.$$

The pairing $V^\vee \times V \to K : (f, v) \mapsto f(v)$ can thus in coordinates be written like the standard Euclidean scalar product (or matrix multiplication $(\mu_1, \ldots, \mu_n) \cdot \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix}$).

2.9 Definition. Let $f \in \mathrm{Hom}_K(V, W)$. We define its dual map:

$$f^t : W^\vee \to V^\vee : g \mapsto f^t(g) := g \circ f$$

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[rr, bend right, "f^t(g)"'] & W \arrow[r, "g"] & K \end{tikzcd}$$

2.10 Lemma. *Let $f, \tilde{f} \in \mathrm{Hom}(V, W)$, $f' \in \mathrm{Hom}(W, U)$, $\lambda \in K$. Them:*

(1) f^t *is linear*

(2) $(\mathrm{id}_V)^t = \mathrm{id}_{V^\vee}$

(3) $(f' \circ f)^t = f^t \circ (f')^t$

(4) f isomorphism $\Rightarrow f^t$ isomorphism

(5) $(f+\tilde{f})^t = f^t + \tilde{f}^t$, $(\lambda f)^t = \lambda f^t$. In particular

$$t : \mathrm{Hom}_K(V,W) \to \mathrm{Hom}_K(W^\vee, V^\vee),\ f \mapsto f^t$$

is a linear map.

Beweis. (1) Let $g, h \in W^\vee$, $\lambda \in K$. Then

$$\begin{aligned} f^t(g+h) &= (g+h)\circ f = g\circ f + h \circ f = f^t(g) + f^t(h), \\ f^t(\lambda g) &= (\lambda g)\circ f = \lambda(g \circ f) = \lambda f^t(g). \end{aligned}$$

(2) Let $g \in V^\vee$, then

$$(\mathrm{id}_V)^t(g) = g \circ \mathrm{id}_V = g \quad \Rightarrow \quad (\mathrm{id}_V)^t = \mathrm{id}_{V^\vee}.$$

(3) $V \xrightarrow{\text{f}} W \xrightarrow{\text{f'}} U$, $f' \circ f \in \mathrm{Hom}(V,U)$. Let $g \in U^\vee$, then

$$\begin{aligned} (f'\circ f)^t(g) = g \circ f' \circ f = (g\circ f')\circ f = f^t(g\circ f') = f^t((f')^t(g)) \\ \Rightarrow (f' \circ f)^t = f^t \circ (f')^t. \end{aligned}$$

(4) If f is an isomorphism, there exists $f^{-1} : W \to V$. Using 2) and 3) we have

$$\begin{aligned} (f^{-1})^t \circ f^t &= (f \circ f^{-1})^t = (\mathrm{id}_W)^t = \mathrm{id}_{W^\vee} \\ f^t \circ (f^{-1})^t &= (f^{-1}\circ f)^t = (\mathrm{id}_V)^t = \mathrm{id}_{V^\vee}, \end{aligned}$$

thus f^t is invertible with inverse $(f^{-1})^t$.

(5) Follows via computation.

□

2.11 Theorem (Dual maps and transposed). *Let V, W be vector spaces over K with bases $B = (b_1, \dots, b_n)$, $C = (c_1, \dots, c_m)$, $f \in \mathrm{Hom}_K(V,W)$. Then*

$$({}_BM_C(f))^T = {}_{C^\vee}M_{B^\vee}(f^t)$$

Beweis. Let $x \in W$, $x = \sum_{i=1}^m \mu_i c_i$. Then with $f : V \to W$, $f^t : W^\vee \to V^\vee : g \mapsto g \circ f$, $V \xrightarrow{\text{f}} W \xrightarrow{\text{g}} K$

$$c_i^\vee(x) = c_i^\vee(\sum_j \mu_j c_j) = \sum_j \mu_j c_i^\vee(c_j) = \sum_j \mu_j \delta_{ij} = \mu_i,$$

thus $x = \sum_{i=1}^m c_i^\vee(x) c_i$. For $f(b_j) \in W$ we thus have $f(b_j) = \sum_{i=1}^m c_i^\vee(f(b_j))c_i$.
Hence ${}_BM_C(f) = (c_i^\vee(f(b_j)))_{i=1,\dots,m,\ j=1,\dots,n}$. For $g \in V^\vee$ we have $g = \sum_{i=1}^n g(b_i)b_i^\vee$, since the values of these maps coincide on the basis B. In particular,

$$\begin{aligned} f^t(c_i^\vee) = \sum_{j=1}^n (f^t(c_i^\vee)(b_j))b_j^\vee = \sum_{j=1}^n (c_i^\vee \circ f)(b_j) b_j^\vee = \sum_{j=1}^n c_i^\vee(f(b_j))b_j^\vee \\ \Rightarrow {}_{C^\vee}M_{B^\vee}(f^t) = (c_i^\vee(f(b_j)))_{i=1,\dots,m,\ j=1,\dots,n} = {}_BM_C(f)^T \end{aligned}$$

□

2.12 Remark. The following diagram is commutative:

$$
\begin{tikzcd}
f \arrow[rrr, mapsto] \arrow[dddd] \arrow[dr, phantom, "\in"] & & & f^t \arrow[dddd] \\
& \mathrm{Hom}_K(V,W) \arrow[r, "t"'] \arrow[dd] & \mathrm{Hom}_K\left(W^\vee, V^\vee\right) \arrow[dd] \arrow[ur, phantom, "\ni"] & \\
& & & \\
& \mathrm{Mat}(m \times n, K) \arrow[r, "\mathrm{T}"] \arrow[dl, phantom, "\in"] & \mathrm{Mat}(n \times m, K) \arrow[dr, phantom, "\ni"] & \\
{}_B M_C(f) \arrow[rrr, mapsto] & & & {}_{C^\vee} M_{B^\vee}(f^t) \\
& A \arrow[r, mapsto] & A^T &
\end{tikzcd}
$$

In particular: $t : \mathrm{Hom}_K(V, W) \xrightarrow{\cong} \mathrm{Hom}_K(V^\vee, W^\vee)$ is an isomorphism.

2.13 Definition (Annihilator). Let $U \subset V$ be a linear subspace.

$$U^0 = \{g \in V^\vee \mid g(u) = 0 \,\forall\, u \in U\} \subset V^\vee$$

is the **annihilator** of U.

2.14 Remark. Annihilators are subspaces.

2.15 Theorem (Dimension annihilator). *Let $U \subset V$ be a subspace. Then $\dim U^0 = \dim V - \dim U$. If $(u_1, \ldots, u_k)$ is a basis of U and $(u_1, \ldots, u_k, v_1, \ldots, v_r)$ is a basis of V, then $(v_1^\vee, \ldots, v_r^\vee)$ is a basis of U^0.*

Beweis. $v_1^\vee, \ldots, v_r^\vee$ are linearly independent, since they belong to the same basis. It remains to show: $\langle v_1^\vee, \ldots, v_r^\vee \rangle = U^0$
" $\supset$ ": Let $g \in U^0$, $g = \mu_1 u_1^\vee + \ldots + \mu_k u_k^\vee + \lambda_1 v_1^\vee + \ldots + \lambda_r v_r^\vee$. We have

$$0 = g(u_i) = \mu_i \quad \Rightarrow \quad g \in \langle v_1^\vee, \ldots, v_r^\vee \rangle.$$

" $\subset$ ": Since $v_j^\vee(u_i) = 0 \,\forall\, i = 1, \ldots, k$ we have $v_j^\vee \in U^0 \;\forall\, j = 1, \ldots, r$. □

2.16 Theorem (Annihilators and dual maps). *Let $f \in \mathrm{Hom}_K(V, W)$.*

(1) $\mathrm{Ker}(f^t) = (\mathrm{Im}(f))^\circ$

(2) $\mathrm{Im}(f^t) = (\mathrm{Ker(f)})^\circ$

Beweis. (1) $f^t \in \mathrm{Hom}_K(W^\vee, V^\vee)$, let $g \in W^\vee$, then

$$
\begin{aligned}
g \in \mathrm{Ker}(f^t) &\Leftrightarrow 0 = f^t(g) = g \circ f \in V^\vee \\
&\Leftrightarrow g \circ f(x) = 0 \,\forall\, x \in V \\
&\Leftrightarrow g(y) = 0 \,\forall\, y \in \mathrm{Im}(f) \subset W \\
&\Leftrightarrow g \in (\mathrm{Im}(f))^\circ.
\end{aligned}
$$

(2) "$\subset$": Let $g \in \mathrm{Im}(f^t) \subset V^\vee \Rightarrow \exists h \in W^\vee$ with $g = f^t(h) = h \circ f$.
Let $x \in \mathrm{Ker}(f) \Rightarrow g(x) = h \circ f(x) = h(0) = 0 \Rightarrow g \in \mathrm{Ker}(f)^\circ$.
"$\supset$": Let $g \in V^\vee$ with $g|_{\mathrm{Ker}(f)} = 0$. We construct $h \in W^\vee$ with $g = f^t(h) = h \circ f$. Choose bases $B = (u_1, \ldots, u_r, v_1, \ldots, v_k)$ of V, $C = (w_1, \ldots, w_r, w_{r+1}, \ldots, w_m)$ of W with $\mathrm{Ker}(f) = \langle v_1, \ldots, v_k \rangle$, $\mathrm{Im}(f) = \langle w_1, \ldots, w_r \rangle$, $f(u_i) = w_i$, $i = 1, \ldots, r$. Set

$$h(w_i) = \begin{cases} g(u_i) & i = 1, \ldots, r \\ 0 & \text{else} \end{cases}$$

Then

$$h \circ f(u_i) = h(f(u_i)) = h(w_i) = g(u_i) \; i = 1, \ldots, r$$

and

$$h \circ f(v_i) = h(0) = 0 = g(v_i) \; i = 1, \ldots, k,$$

since $g|_{\mathrm{Ker}(f)} = 0$.
$\Rightarrow h \circ f = g \Rightarrow f^t(h) = g \Rightarrow g \in \mathrm{Im}(f^t)$.

□

2.17 Corollary. $\mathrm{rank}(f^t) = \mathrm{rank}(f)$

Beweis.

$\mathrm{rank}(f^t) = \dim(\mathrm{Im}(f^t)) = \dim(\mathrm{Ker}(f)^\circ) = \dim V - \dim(\mathrm{Ker}(f)) = \dim(\mathrm{Im}(f)) = \mathrm{rank}(f)$.

□

2.18 Corollary. *Let* $A \in \mathrm{Mat}(m \times n, K)$. *Then* $\mathrm{rowrank}(A) = \mathrm{columnrank}(A)$.

Beweis. $A = {}_EM_E(f_A)$ for the canonical basis E and $f_A : K^n \to K^m : x \mapsto Ax$. By the theorem on dual maps and transposed matrices,

$$\begin{aligned} {}_{E^\vee}M_{E^\vee}(f_A^t) &= ({}_EM_E(f_A))^T = A^T \\ \mathrm{columnrank}(A) = \dim(\mathrm{Im}(f_A)) &= \mathrm{rank}(f_A) \overset{\text{corollary 2.17}}{=} \mathrm{rank}(f_A^t) \\ &= \dim(\mathrm{Im}(f_A^t)) = \mathrm{columnrank}(A^T) = \mathrm{rowrank}(A) \end{aligned}$$

□

2.19 Definition (bilinear form). Let V, W be vector spaces over K.

$$b : V \times W \to K : (v, w) \mapsto b(v, w)$$

is a **bilinear form**, if

$$b_v : W \to K : w \mapsto b(v, w)$$

and

$$b_w : V \to K : v \mapsto b(v, w)$$

are linear $\forall v \in V$, $w \in W$. We then obtain linear maps

$$\begin{aligned} b' &: V \to W^\vee : v \mapsto b_v \\ b'' &: W \to V^\vee : w \mapsto b_w \end{aligned}$$

A bilinear form is called **non-degenerate**, if b' and b'' are injective.

2.20 Example. Scalar products are bilinear forms and are non-degenerate:

$$b' : V \to V^\vee : v \mapsto \langle v, \cdot \rangle$$

Let $w \in \mathrm{Ker}(b') \Rightarrow$ the map $\langle w, \cdot \rangle$ is the zero map $\Rightarrow \langle w, v \rangle = 0 \,\forall v \in V$, in particular $\langle w, w \rangle = 0$.
Since a scalar product is positive definite, $w = 0 \quad \Rightarrow b'$ is injective, b'' analogously.

2.21 Theorem. *Let V, W be finite dimensional, $b : V \times W \to K$ a non-degenerate bilinear form. Then*

$$b' : V \xrightarrow{\cong} W^\vee, \quad b'' : W \xrightarrow{\cong} V^\vee$$

In particular, $\dim V = \dim W$.

Beweis. Since b' is injective, $\dim V \leq \dim W^\vee = \dim W$, since b'' is injective,
$\dim W \leq \dim V^\vee = \dim V \Rightarrow \dim V = \dim W$ and b', b'' are isomorphisms. □

2.22 Corollary. *Let V be a Euclidean vector space with scalar product $\langle \cdot, \cdot \rangle$. Then*

$$V \xrightarrow{\cong} V^\vee : v \mapsto \langle v, \cdot \rangle$$

and

$$V \xrightarrow{\cong} V^\vee : v \mapsto \langle \cdot, v \rangle$$

2.23 Example. Let $V = \mathbb{R}^n$, $\langle \cdot, \cdot \rangle$ the standard scalar product, E the canonical basis. Then

$$V \to V^\vee : v \mapsto \langle v, \cdot \rangle$$

and

$$V \to V^\vee : e_i \mapsto e_i^\vee$$

are the same linear map.

2.24 Theorem (Scalar product and dual space). *Let V be Euclidean,* $\dim V < \infty$.

$$\Psi : V \xrightarrow{\cong} V^\vee : v \mapsto \langle v, \cdot \rangle$$

(1) Let $U \subset V$ be a subspace. Then $\Psi(U^\perp) = U^0$

(2) Let $B = (b_1, \ldots, b_n)$ be an orthonormal basis of V and $B^\vee = (b_1^\vee, \ldots, b_n^\vee)$ the dual basis. Then $\Psi(b_i) = b_i^\vee$.

Beweis. (1) $\dim U^\perp = \dim V - \dim U = \dim U^0$, thus it is enough to show $\Psi(U^\perp) \subset U^0$.
Let $v \in U^\perp \Rightarrow \langle v, u \rangle = 0 \,\forall\, u \in U \Rightarrow \langle v, \cdot \rangle \in U^0$.

(2) $\Psi(b_i) = \langle b_i, \cdot \rangle$ and $\langle b_i, b_j \rangle = \delta_{ij}$ as B ONB. Thus $\langle b_i, \cdot \rangle = b_i^\vee$.

□

2.25 Definition (**Adjoint map**)**.** Let V, W Euclidean and finite dimensional, $f \in \mathrm{Hom}_K(V, W)$. The linear map $f^* : W \to V$, for which $\langle f(v), w\rangle = \langle v, f^*(w)\rangle$ $\forall\, v \in V, w \in W$ is the **adjoint map** of f.

2.26 Lemma (**Adjoint and dual map**)**.** $f^* = \phi^{-1} \circ f^t \circ \Psi$

$$\begin{tikzcd}
v \arrow[d, mapsto] & V \arrow[r, shift right, "f"'] \arrow[d, "\phi"'] & W \arrow[l, shift right, "f^*"'] \arrow[d, "\Psi"] & w \arrow[d, mapsto] \\
\langle\cdot, v\rangle & V^\vee & W^\vee \arrow[l, "f^t"'] & \langle\cdot, w\rangle \\
 & g \circ f & g \arrow[l, mapsto] &
\end{tikzcd}$$

Beweis.

$$f^t(\Psi(w)) = f^t(\langle\cdot, w\rangle) = \langle f(\cdot), w\rangle = \langle\cdot, f^*(w)\rangle = \phi(f^*(w)).$$

□

2.27 Corollary.

$$\mathrm{Im}(f^*) = \mathrm{Ker}(f)^\perp, \mathrm{Ker}(f^*) = \mathrm{Im}(f)^\perp$$

Follows using lemma 2.26, theorem 2.16 and theorem 2.24.

2.28 Remark. In the following, we work in $\mathbb{R}^n$. $\mathbb{R}^n$ is a Euclidean space. We identify its dual space with $\mathbb{R}^n$ using the scalar product. Important is not the dual space itself, but the relation between $\mathbb{R}^n$ and $(\mathbb{R}^n)^\vee$ (they are isomorph).

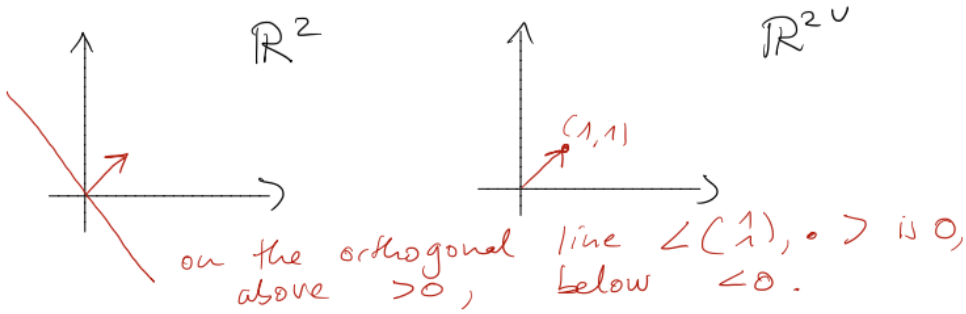


Abbildung 8: Relation between $\mathbb{R}^2$ and $(\mathbb{R}^2)^\vee$ [8]

Convention:
In $\langle\cdot, \cdot\rangle$, we insert in the first place objects $m \in (\mathbb{R}^n)^\vee$, in the second $u \in \mathbb{R}^n$. $m \in (\mathbb{R}^n)^\vee$ defines a linear form on $\mathbb{R}^n$ via $u \mapsto \langle m, u\rangle$. Note that, vice versa, u also defines a linear form on $(\mathbb{R}^n)^\vee$ via $m \mapsto \langle m, u\rangle$. Thus, the dual space of $(\mathbb{R}^n)^\vee$ gets canonically identified with $\mathbb{R}^n$. (See also linear algebra 2.)

[8] Image from Hannah Markwig.

3 Cones, polytopes, polyhedra: the duality theorem

Let $\mathbb{R}^n, (\mathbb{R}^n)^\vee$ be dual vector spaces.

3.1 Definition. A **(convex, polyhedral) cone** in $\mathbb{R}^n$ is of the form

$$\sigma = \mathrm{Cone}(S) = \{\sum_{u \in S} \lambda_u u | \lambda_u \geq 0\}$$

where $S \subset \mathbb{R}^n$ is finite.
We say σ is **generated** by S. We set $\mathrm{Cone}(\emptyset) = \{0\}$.

3.2 Remark. σ is convex, i.e.

$$x, y \in \sigma \Rightarrow \lambda x + (1-\lambda) y \in \sigma \quad \forall\, 0 \leq \lambda \leq 1,$$

σ is a cone: $x \in \sigma \Rightarrow \lambda x \in \sigma \;\forall\, \lambda \geq 0$.

3.3 Example. Some cones:

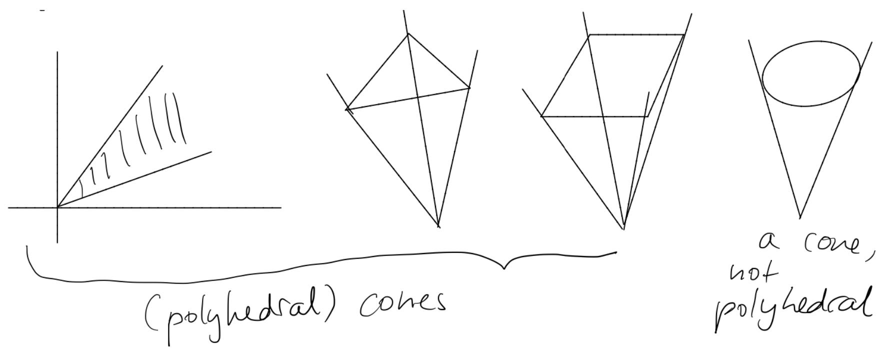


Abbildung 9: Cones [9]

3.4 Example. $\mathrm{Cone}(e_1, -e_1) \subset \mathbb{R}^2$
$\mathrm{Cone}(e_1, -e_1, e_2) \subset \mathbb{R}^2$

3.5 Definition. A **polytope** is a set of the form

$$P = \mathrm{Conv}(S) = \{\sum_{u \in S} \lambda_u u | \lambda_u \geq 0, \sum \lambda_u = 1\} \subset \mathbb{R}^n$$

where $S \subset \mathbb{R}^n$ is finite. We say P is the **convex hull** of S.

3.6 Remark. One can define convex hulls as

$$\mathrm{Conv}(K) = \bigcap_{K' \subset \mathbb{R}^n,\, K' \text{ convex},\, K \subset K'} K'$$

i.e. the smallest convex set containing K. Then we can show:

$$\mathrm{Conv}(K) = \{\lambda_1 x_1 + \ldots + \lambda_k x_k, \{x_1, \ldots, x_k\} \subset K, \lambda_i \geq 0, \sum \lambda_i = 1\}$$

[9]Image from Hannah Markwig.

in the usual way:
" $\supset$ ": Induction on k. For $k=2$, $\lambda_1 x_1 + \lambda_2 x_2 \in \operatorname{Conv}(K)$ for $x_1, x_2 \in K$, $\lambda_i \geq 0$, $\lambda_1 + \lambda_2 = 1$, since $\operatorname{Conv}(K)$ is convex. For the induction step, consider $\lambda_1 x_1 + \ldots + \lambda_k x_k$ and assume $\lambda_k \neq 1$.

$$\lambda_1 x_1 + \ldots + \lambda_k x_k = (1-\lambda_k)\left(\frac{\lambda_1}{1-\lambda_k} x_1 + \ldots + \frac{\lambda_{k-1}}{1-\lambda_k} x_{k-1}\right) + \lambda_k x_k,$$

but since

$$\begin{aligned}
&\lambda_1 + \ldots + \lambda_k = 1 \\
\Rightarrow &\lambda_1 + \ldots + \lambda_{k-1} = 1 - \lambda_k \\
\Rightarrow &\frac{\lambda_1}{1-\lambda_k} + \ldots + \frac{\lambda_{k-1}}{1-\lambda_k} = 1 \\
\Rightarrow &\frac{\lambda_1}{1-\lambda_k} x_1 + \ldots + \frac{\lambda_{k-1}}{1-\lambda_k} x_{k-1} \in \operatorname{Conv}(K)
\end{aligned}$$

by induction assumption, using convexity we obtain again $\lambda_1 x_1 + \ldots + \lambda_k x_k \in \operatorname{Conv}(K)$
" $\subset$ ": We show the right hand side is convex:
let $\lambda_1 x_1 + \ldots + \lambda_k x_k$, $\mu_1 y_1 + \ldots + \mu_l y_l$ in the right hand side. Then

$$\lambda(\lambda_1 x_1 + \ldots + \lambda_k x_k) + (1-\lambda)(\mu_1 y_1 + \ldots + \mu_l y_l)$$

has coefficients $\lambda\lambda_i$, $(1-\lambda)\mu_j$ and

$$\sum \lambda\lambda_i + \sum (1-\lambda)\mu_j = \lambda \sum \lambda_i + (1-\lambda) \sum \mu_j = \lambda \cdot 1 + (1-\lambda) * 1 = 1,$$

thus this sum is also in the right hand side. Thus the right hand side is a convex set containing K, and hence $\operatorname{Conv}(K) \subset$ right hand side.

3.7 Remark. If $K = \{x_1, \ldots, x_k\}$ is finite, the right hand side in remark 3.6 equals $\{\sum \lambda_i x_i | \lambda_i \geq 0, \sum \lambda_i = 1\}$ as used in definition 3.5.

3.8 Remark. We can compare:

Linear Geometry	affine geometry
cones	polytopes
have a 0-point	no special point
$\sigma = \{\lambda \cdot (u,1) \in \mathbb{R}^{n+1} \| u \in P, \lambda \geq 0\}$	P
$\sigma = \operatorname{Cone}(S)$ s.th. $\forall\, u \in S : u_{n+1} > 0$	$\sigma \cap \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} + \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_n \\ 0 \end{pmatrix} \| x_i \in \mathbb{R} \right\}$

Figure 10 shows a polytope in $\mathbb{R}^{n+1}$.

3.9 Definition. Let $\sigma \subset \mathbb{R}^n$ be a cone. $\dim \sigma :=$ dimension of the smallest linear subspace $U = \operatorname{span}(\sigma) \subset \mathbb{R}^n$ containing σ.
($P \subset \mathbb{R}^n$ a polytope, $\dim P :=$ dimension of the smallest affine subspace $W = \operatorname{aff}(P) \subset \mathbb{R}^n$ containing P.)

3.10 Definition (dual cone). $\sigma \subset \mathbb{R}^n$ a cone.
The **dual cone** is $\sigma^\vee = \{m \in (\mathbb{R}^n)^\vee | \langle m, u \rangle \geq 0 \,\forall\, u \in \sigma\}$
We will see later why this is a cone.

[10]Image from Hannah Markwig.

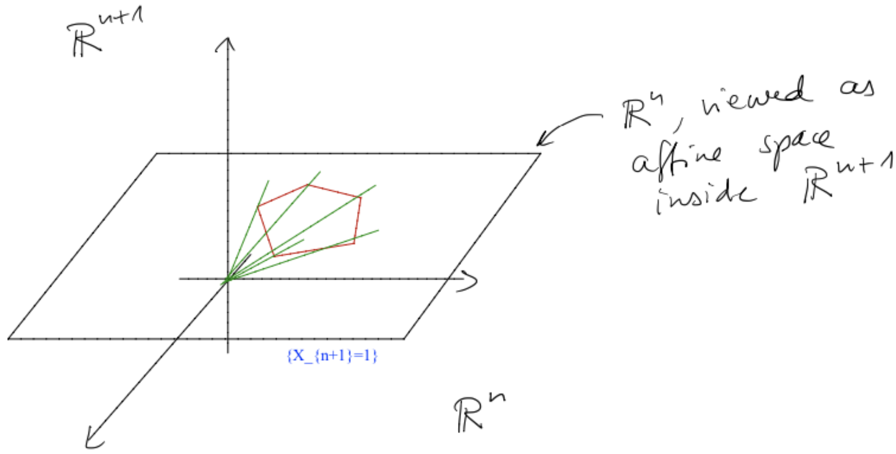


Abbildung 10: Polytope in $\mathbb{R}^{n+1}$ [10]

3.11 Example. $\sigma = \text{Cone}\left(\begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}-1\\1\end{pmatrix}\right)$

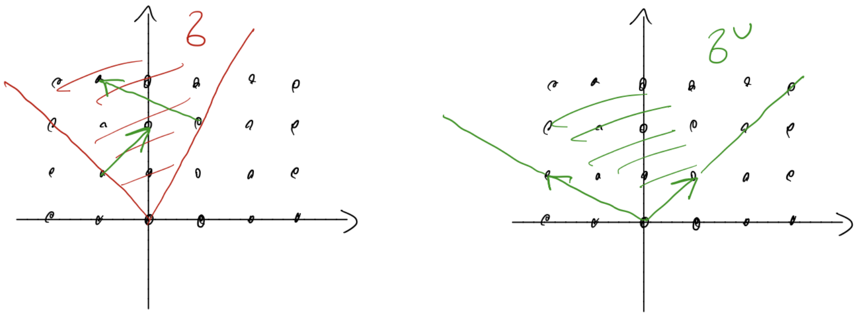


Abbildung 11: σ and $\sigma^\vee$ [11]

$\sigma^\vee = \text{Cone}\left(\begin{pmatrix}-2\\1\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}\right)$:

We have

$$\left\langle \begin{pmatrix}-2\\1\end{pmatrix}, \begin{pmatrix}1\\2\end{pmatrix} \right\rangle = 0, \quad \left\langle \begin{pmatrix}-2\\1\end{pmatrix}, \begin{pmatrix}-1\\1\end{pmatrix} \right\rangle = 3 > 0$$

For any element u in σ, we have $u = \lambda_1 \begin{pmatrix}1\\2\end{pmatrix} + \lambda_2 \begin{pmatrix}-1\\1\end{pmatrix}$, $\lambda_1, \lambda_2 \geq 0$

$$\begin{aligned}
\Rightarrow \left\langle \begin{pmatrix}-2\\1\end{pmatrix}, u \right\rangle &= \left\langle \begin{pmatrix}-2\\1\end{pmatrix}, \lambda_1 \begin{pmatrix}1\\2\end{pmatrix} + \lambda_2 \begin{pmatrix}-1\\1\end{pmatrix} \right\rangle \\
&= \lambda_1 \left\langle \begin{pmatrix}-2\\1\end{pmatrix}, \begin{pmatrix}1\\2\end{pmatrix} \right\rangle + \lambda_2 \left\langle \begin{pmatrix}-2\\1\end{pmatrix}, \begin{pmatrix}-1\\1\end{pmatrix} \right\rangle \\
&= 3\lambda_2 \geq 0
\end{aligned}$$

[11] Image from Hannah Markwig.

Analogously: $\left\langle \begin{pmatrix} 1 \\ 1 \end{pmatrix}, u \right\rangle \geq 0 \, \forall \, u \in \sigma$

$$\Rightarrow \langle m, u \rangle \geq 0 \, \forall \, m \in \text{Cone}\left(\begin{pmatrix} -2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right)$$
$$\Rightarrow \text{Cone}\left(\begin{pmatrix} -2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) \subset \sigma^\vee$$

To see equality, take a point m outside $\text{Cone}\left(\begin{pmatrix} -2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right)$.
Since $\begin{pmatrix} -2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ are linearly independent, they form a basis of $\mathbb{R}^2$ and thus $\exists! \, \lambda_1, \lambda_2$:

$$m = \lambda_1 \begin{pmatrix} -2 \\ 1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Since $m \notin \text{Cone}\left(\begin{pmatrix} -2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right)$, at least one of the coefficients are negative.
Assume without restriction $\lambda_1 < 0$. Then

$$\left\langle m, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\rangle = \left\langle \lambda_1 \begin{pmatrix} -2 \\ 1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\rangle = 3\lambda_1 < 0$$
$$\Rightarrow m \notin \sigma^\vee$$
$$\Rightarrow \text{Cone}\left(\begin{pmatrix} -2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right)^{\mathrm{C}} \subset (\sigma^\vee)^{\mathrm{C}}$$
$$\Rightarrow \sigma^\vee \subset \text{Cone}\left(\begin{pmatrix} -2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right)$$
$$\Rightarrow \sigma^\vee = \text{Cone}\left(\begin{pmatrix} -2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right)$$

3.12 Example. $\sigma = \text{Cone}\left(\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right) \subset \mathbb{R}^3$

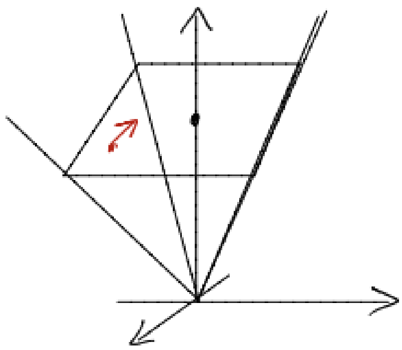


Abbildung 12: σ [12]

[12] Image from Hannah Markwig.

What are the vectors orthogonal to the facets of this cone?

Plane through 0, $\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} : \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ is orthogonal.
Plane through 0, $\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} : \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ is orthogonal.
Plane through 0, $\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} : \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$ is orthogonal.
Plane through 0, $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} : \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ is orthogonal.

In $(\mathbb{R}^3)^\vee$, in the affine space $\{x_3 = 1\}$:

Abbildung 13: Sketch of $\sigma^\vee$ [13]

One can prove, similar to example 3.11 that this is indeed the dual cone.

3.13 Definition. For $m \in (\mathbb{R}^n)^\vee \setminus \{0\}$ we define the hyperplane

$$H_m = \{u \in \mathbb{R}^n | \langle m, u\rangle = 0\}$$

and the closed halfspace

$$H_m^+ = \{u \in \mathbb{R}^n | \langle m, u\rangle \geq 0\}$$

For a cone σ, H_m is called a **supporting hyperplane** if $\sigma \subset H_m^+$.
H_m^+ is then called **supporting halfspace**.

3.14 Lemma. *H_m is supporting hyperplane for σ $\Leftrightarrow$ $m \in \sigma^\vee \setminus \{0\}$.*

Beweis.

$$\begin{aligned} & H_m \text{ is supporting hyperplane for } \sigma \\ \Leftrightarrow\, & \sigma \subset H_m^+ \\ \Leftrightarrow\, & \sigma \subset \{u \in \mathbb{R}^n | \langle m, u\rangle \geq 0\}, m \neq 0 \\ \Leftrightarrow\, & \langle m, u\rangle \geq 0 \,\forall\, u \in \sigma, m \neq 0 \\ \Leftrightarrow\, & m \in \sigma^\vee \setminus \{0\} \end{aligned}$$

□

[13] Image from Hannah Markwig.

3.15 Lemma. *Let σ be a cone, $u \notin \sigma \Rightarrow \exists$ seperating hyperplane, i.e.* $\exists\, m \in \sigma^\vee : \langle m, u \rangle < 0$.
Without proof, only picture 14: (methodes from analysis)

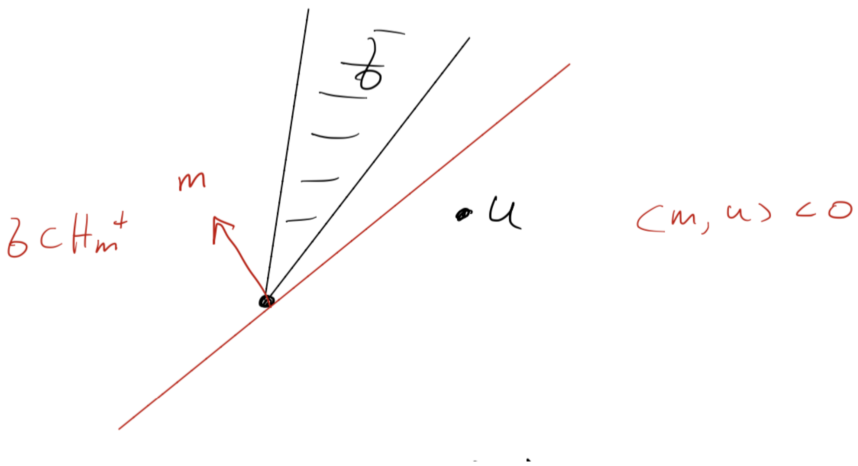


Abbildung 14: Seperating hyperplane [14]

3.16 Lemma. $(\sigma^\vee)^\vee = \sigma$

Beweis. " $\supset$ ": Let $u \in \sigma \Rightarrow \langle m, u \rangle \geq 0 \,\forall\, m \in \sigma^\vee \Rightarrow u \in (\sigma^\vee)^\vee$
" $\subset$ ": Assume there was $u \in (\sigma^\vee)^\vee$ with $u \notin \sigma$. Then there exists a seperating hyperplane, i.e. $m \in \sigma^\vee$ with $\langle m, u \rangle < 0$ but this contradicts $u \in (\sigma^\vee)^\vee$, ↯. □

If $\sigma^\vee$ is a cone, $\sigma^\vee = \operatorname{Cone}(m_1, \ldots, m_r)$, then

$$\begin{aligned}\sigma = (\sigma^\vee)^\vee &= \{u \in \mathbb{R}^n | \langle m, u \rangle \geq 0 \,\forall\, m \in \sigma^\vee\} \\ &= \{u \in \mathbb{R}^n | \langle m_i, u \rangle \geq 0,\, i = 1, \ldots, r\} \\ &= H_{m_1}^+ \cup \ldots \cup H_{m_r}^+\end{aligned}$$

We would thus have: σ is the intersection of finitely many closed halfspaces. Vice versa, if $\sigma = H_{m_1}^+ \cup \ldots \cup H_{m_r}^+$ then $\sigma^\vee = \operatorname{Cone}(m_1, \ldots, m_r)$ is a cone.

Our next (non trivial!) goal is to show that every cone is the intersection of finitely many halfspaces. We treat this in more generality:

3.17 Definition. A **polyhedron** in $\mathbb{R}^n$ is the solution set of a system of linear equalities and inequalities:

$$P = \{x \in \mathbb{R}^n | Bx = c, Ax \leq b\}.$$

3.18 Example. In $\mathbb{R}^3$,

$$B = (0\ 0\ 1), \quad c = 1, \quad A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & 1 & 0 \end{pmatrix} \quad b = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

[14]Image from Hannah Markwig.

$$B\begin{pmatrix} x \\ y \\ z \end{pmatrix} = c \quad \Leftrightarrow \quad z = 1$$

$$A\begin{pmatrix} x \\ y \\ z \end{pmatrix} \leq b \quad \Leftrightarrow \quad -x \leq 0, -y \leq 0, x + y \leq 1$$

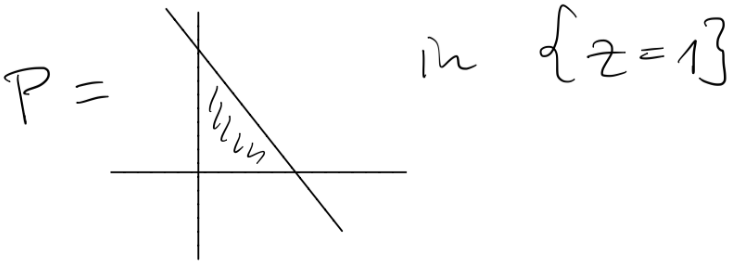


Abbildung 15: P in $\{z = 1\}$ [15]

3.19 Definition. The **Minkowsi sum** of two subsets $P, Q \subset \mathbb{R}^n$ is $P + Q = \{p + q | p \in P, q \in Q\}$

3.20 Example. (1) In $\mathbb{R}^3$, $\operatorname{Conv}\left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right) = P$,

$$\operatorname{Conv}\left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right) = Q$$

Abbildung 16: $P + Q$ [16]

(2) In $\mathbb{R}^2$, $\operatorname{Conv}\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) = P$, $\operatorname{Conv}\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) = Q$

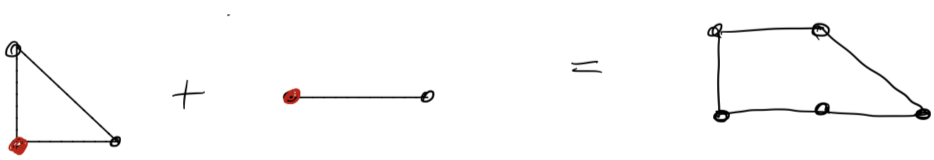


Abbildung 17: $P + Q$ [17]

[15] Image from Hannah Markwig.
[16] Image from Hannah Markwig.
[17] Image from Hannah Markwig.

(3) Figure 18

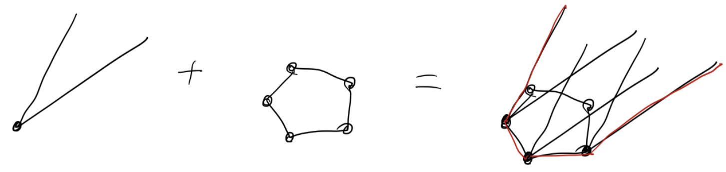


Abbildung 18: $P + Q$ [18]

3.21 Theorem. *[**dual description of polytopes, cones and polyhedra**].*

(1) <u>Polytopes:</u> $P \in \mathbb{R}^n$ is a polytope $\Leftrightarrow$ P is a bounded polyhedron.

(2) <u>Cones:</u> $\sigma \in \mathbb{R}^n$ is a cone $\Leftrightarrow$ σ is the intersection of a linear subspace with finitely many closed halfspaces, i.e. inside $\operatorname{span}(\sigma)$ *we can write $\sigma = H^+_{m_1} \cap \ldots \cap H^+_{m_r}$.*

(3) <u>Polyhedra:</u> $P = \operatorname{Conv}(V) + \operatorname{Cone}(Y)$ for $V, Y \subset \mathbb{R}^n$ finite $\Leftrightarrow$ P is a polyhedron.

Beweis. We will prove 2) later. Now: "2) $\Rightarrow$ 3)"
" $\Leftarrow$ ": Assume P is a polyhedron. $P = \{x | Bx = c, Ax \leq b\}$. We can work in the affine space $Bx = c$, that is, we can assume without restriction $P = \{x | Ax \leq b\}$. Let

$$\sigma = \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_n \\ x_{n+1} \end{pmatrix} \middle| \begin{pmatrix} & \text{A} & & -b_1 \\ & & & \vdots \\ & & & -b_n \\ 0 & \ldots & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \\ x_{n+1} \end{pmatrix} \leq 0 \right\}$$

i.e. if P is given by the inequalities $a_i x \leq b_i$ ($x = (x_1, \ldots, x_n)$, a_i = rows of A), then σ is the intersection of the halfspaces $x_{n+1} \geq 0$, $a_i x - b_i x_{n+1} \leq 0 \Leftrightarrow a_i x \leq b_i x_{n+1}$. Thus we have

$$P = \left\{ x \in \mathbb{R}^n | \begin{pmatrix} x \\ 1 \end{pmatrix} \in \sigma \right\}, \quad \sigma \cap \{x_{n+1}\} = P$$

Now σ is the intersection of finitely many halfspaces $\overset{2)}{\Rightarrow}$ σ is a cone, $\sigma = \operatorname{Cone}(w_1, \ldots, w_r)$ $w_i \in \mathbb{R}^{n+1}$, $(w_i)_{n+1} \geq 0$ (as $\sigma \subset \{x_{n+1} \geq 0\}$). Without restriction, we can replace any w_i with $(w_i)_{n+1} > 0$ with the intersection of the ray $\{\lambda w_i | \lambda \geq 0\}$ with $\{x_{n+1} = 1\}$, i.e. without restriction $(w_i)_{n+1} = 1$.
Let $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ be the projection. <u>claim:</u>

$$P = \operatorname{Conv}(\pi(w_i) | (w_i)_{n+1} = 1) + \operatorname{Cone}(\pi(w_i) | (w_i)_{n+1} = 0)$$

" $\subset$ ": Let $x \in P \Rightarrow \begin{pmatrix} x \\ 1 \end{pmatrix} \in \sigma \Rightarrow \begin{pmatrix} x \\ 1 \end{pmatrix} = \lambda_1 w_1 + \ldots + \lambda_r w_r$, $\lambda_i \geq 0$.
Without restriction, let

$$(w_1)_{n+1} = \ldots = (w_s)_{n+1} = 1 \quad \text{and} \quad (w_{s+1})_{n+1} = \ldots = (w_r)_{n+1} = 0$$

[18]Image from Hannah Markwig.

$$x = \underbrace{\lambda_1 \pi(w_1) + \ldots + \lambda_s \pi(w_s)}_{\in \mathrm{Conv}(\pi(w_1),\ldots,\pi(w_s))} + \underbrace{\lambda_{s+1}\pi(w_{s+1}) + \ldots + \lambda_r \pi(w_r)}_{\in \mathrm{Cone}(\pi(w_{s+1}),\ldots,\pi(w_r))}$$

since $\lambda_1 + \ldots + \lambda_s = 1 = (n+1)$st coordinate.
" $\supset$ ": Let

$$\begin{aligned}
& x \in \mathrm{Conv}(\pi(w_i) | (w_i)_{n+1} = 1) + \mathrm{Cone}(\pi(w_i)|(w_i)_{n+1} = 0) \\
& \Rightarrow x = \lambda_1 \pi(w_1) + \ldots + \lambda_s \pi(w_s) + \lambda_{s+1}\pi(w_{s+1}) + \ldots + \lambda_r \pi(w_r) \\
& \quad \text{with } \lambda_1 + \ldots + \lambda_s = 1,\ \lambda_i \geq 0 \\
& \Rightarrow \begin{pmatrix} x \\ 1 \end{pmatrix} = \lambda_1 w_1 + \ldots + \lambda_s w_s + \lambda_{s+1} w_{s+1} + \ldots + \lambda_r w_r \in \sigma \\
& \Rightarrow x \in P
\end{aligned}$$

" $\Rightarrow$ ": Now assume $P = \mathrm{Conv}(V) + \mathrm{Cone}(Y)$. Let $V = \{w_1, \ldots, w_s\}, Y = \{w_{s+1}, \ldots, w_r\}$ and

$$\sigma = \mathrm{Cone}\left(\begin{pmatrix} w_1 \\ 1 \end{pmatrix}, \ldots, \begin{pmatrix} w_s \\ 1 \end{pmatrix}, \begin{pmatrix} w_{s+1} \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} w_r \\ 0 \end{pmatrix}\right)$$

$\sigma \cap \{x_{n+1} = 1\} = P = \{x | \begin{pmatrix} x \\ 1 \end{pmatrix} \in \sigma\}$, so for any $x \in P$ we have $\begin{pmatrix} x \\ 1 \end{pmatrix} \in \sigma$ and vice versa.
By 2), σ is a finite intersection of halfspaces. Then $P = \{x \in \mathbb{R}^n | \begin{pmatrix} x \\ 1 \end{pmatrix} \in \sigma\}$ is a polyhedron, as we can see by plugging in $\begin{pmatrix} x \\ 1 \end{pmatrix}$ into the inequalities for σ.

"3) $\Rightarrow$ 1)"
" $\Rightarrow$ ": Let P be a polytope, then $P = \mathrm{Conv}(V) + \mathrm{Cone}(Y)$ with $Y = \emptyset \overset{3)}{\Rightarrow} P$ is a polyhedron. Obviously, P is bounded.
" $\Leftarrow$ ": Let P be a bounded polyhedron $\overset{3)}{\Rightarrow} P = \mathrm{Conv}(V) + \mathrm{Cone}(Y)$. If $Y \neq \emptyset$, P was not bounded $\Rightarrow Y = \emptyset \Rightarrow P = \mathrm{Conv}(V)$ is a polytope. □

Now we move on to preparations for the proof on 2). Idea: If

$$\begin{aligned}
\sigma &= \mathrm{Cone}(y_1, \ldots, y_r) \\
&= \{t_1 y_1 + \ldots + t_r y_r,\ t_i \geq 0\} \\
&= \pi(\{(x,t) \in \mathbb{R}^{n \times r} | t_i \geq 0,\ x = Yt\})
\end{aligned}$$

where Y is the matrix with the y_i as columns. The set $\{(x,t)|t_i \geq 0,\ x = Yt\}$ is a polyhedron. We want to understand projections of polyhedra.

3.22 Definition. Let P be a polyhedron

$$\begin{aligned}
\mathrm{proj}_k(P) &:= \pi_k(P) \text{ is its image under the coordinate} \\
&\qquad \text{projection setting the } k\text{-th coordinate to } 0 \\
&= \{x - x_k e_k | x \in P\} \\
&= \{x \in \mathbb{R}^n | x_k = 0,\ \exists\, y \in \mathbb{R} : x + y e_k \in P\} \subset \{x_k = 0\} \subset \mathbb{R}^n \\
\mathrm{elim}_k(P) &:= \pi_k^{-1}(\pi_k(P)) \\
&= \{x - t e_k | x \in P, t \in P\} \\
&= \{x \in \mathbb{R}^n |\, \exists\, y \in \mathbb{R} : x + y e_k \in P\}
\end{aligned}$$

We have $\mathrm{elim}_k(P) = \mathrm{proj}_k(P) \times \mathbb{R}$.

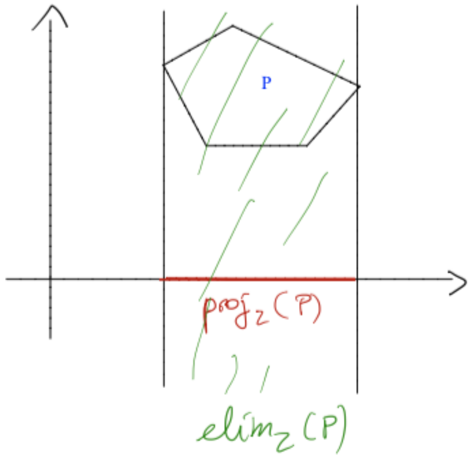


Abbildung 19: $\mathrm{elim}_k(P)$ and $\mathrm{proj}_k(P)$ [19]

Task: Find equations for eliminating and proj.

3.23 Example.

(1) $-x_1 - 4x_2 \leq -9$	(green)
(2) $-2x_1 - x_2 \leq -4$	(purple)
(3) $x_1 - 2x_2 \leq 0$	(pink)
(4) $x_1 \leq 4$	(red)
(5) $2x_1 + x_2 \leq 11$	(orange)
(6) $-2x_1 + 6x_2 \leq 17$	(yellow)
(7) $-6x_1 - x_2 \leq -6$	(blue)

Visualized in Figure 20.

For fixed x_1, what are the possible values of x_2?
Because of (4), we must take $x_1 \leq 4$. Except (4), we can write all the inequalities such that they give upper or lower bounds for x_2:

$$
\begin{aligned}
&(1)\ -x_1 - 4x_2 \leq -9 && \Rightarrow \frac{1}{4}(9 - x_1) \leq x_2 \\
&(2)\ -2x_1 - x_2 \leq -4 && \Rightarrow (-2x_1 + 4) \leq x_2 \\
&(3)\ x_1 - 2x_2 \leq 0 && \Rightarrow \frac{1}{2}x_1 \leq x_2 \\
&(4)\ x_1 \leq 4 && \\
&(5)\ 2x_1 + x_2 \leq 11 && \Rightarrow -2x_1 + 11 \geq x_2 \\
&(6)\ -2x_1 + 6x_2 \leq 17 && \Rightarrow \frac{1}{6}(2x_1 + 17) \geq x_2 \\
&(7)\ -6x_1 - x_2 \leq -6 && \Rightarrow -6x_1 + 6 \leq x_2
\end{aligned}
$$

[19] Image from Hannah Markwig.

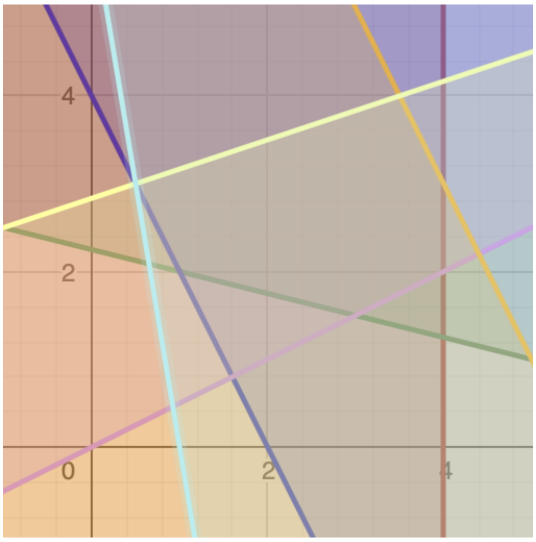


Abbildung 20: Lines (1) to (7)

Choose an upper and lower bound, e.g. $x_2 \geq \frac{1}{4}(9 - x_1)$ and $-2x_1 + 11 \geq x_2$.
In order to have x_2 which satisfies both we must have $-2x_1 + 11 \geq \frac{1}{4}(9 - x_1)$.
If for every pair of upper and lower bound we have the expression in x_1 for the lower bound $\leq$ the expression in x_1 for the upper bound. Then we find x_2 in P. In this way, we can produce inequalities for elim_2 and proj_2! Concretely:

$$\begin{aligned}
&x_1 \leq 4, & \\
&\frac{1}{4}(9 - x_1) \leq -2x_1 + 11, &&\Rightarrow \frac{1}{4}(9 - x_1) \leq \frac{1}{6}(2x_1 + 17) \\
&(-2x_1 + 4) \leq -2x_1 + 11, &&\Rightarrow (-2x_1 + 4) \leq \frac{1}{6}(2x_1 + 17) \\
&\frac{1}{2}x_1 \leq -2x_1 + 11, &&\Rightarrow \frac{1}{2}x_1 \leq \frac{1}{6}(2x_1 + 17) \\
&-6x_1 + 6 \leq -2x_1 + 11, &&\Rightarrow -6x_1 + 6 \leq \frac{1}{6}(2x_1 + 17) \\
&\Leftrightarrow x_1 \leq 4, & \\
&7x_1 \leq 35, &&-14 \leq 14x_1 \\
&4 \leq 11, &&-7 \leq 14x_1 \\
&5x_1 \leq 22, &&2x_1 \leq 34 \\
&-5 \leq 4x_1, &&19 \leq 38x_1
\end{aligned}$$

$$\Leftrightarrow x_1 = 4, \frac{-5}{4} \leq x_1, -1 \leq x_1, \frac{-1}{2} \leq x_1, \frac{1}{2} \leq x_1 \quad \Leftrightarrow \quad \frac{1}{2} \leq x_1 \leq 4$$

In figure 20, the leftmost point of our intersection of halfspaces is the common intersection point of the lines (2), (6) and (7). It has x-coordinate $\frac{1}{2}$:

$$\begin{aligned}
-2x_1 - x_2 &= -4 \\
-6x_1 - x_2 &= -6 \\
-2x_1 + 6x_2 &= 17
\end{aligned}$$

$$\left(\begin{array}{cc|c} -2 & -1 & -4 \\ -6 & -1 & -6 \\ -2 & 6 & 17 \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|c} -2 & -1 & -4 \\ 0 & 2 & 6 \\ 0 & 7 & 21 \end{array}\right)$$

$$\begin{aligned} &\Rightarrow x_2 = 3,\ -2x_1 - 3 = -4 \\ &\Rightarrow -2x_1 = -1 \Rightarrow x_1 = \frac{1}{2} \end{aligned}$$

Now more general. Let σ be the intersection of finitely many halfspaces, $\sigma = H^+_{-a_1} \cap \ldots \cap H^+_{-a_r}$. Write a_i in the rows of a matrix A, then $\sigma = \{x | Ax \leq 0\}$.
Note: If σ is contained in a linear subspace, we can combine two inequalities $\langle a, x\rangle \leq 0$ and $\langle -a, x\rangle \leq 0$ to produce $\langle a, x\rangle = 0$. We can write all equations of our linear space as pairs of two such inequalities into our matrix. Therefore we can restrict to the case $\sigma = \{x|Ax \leq 0\}$ even is σ is contained in a linear subspace.

Let $A^{/k} := \{a_i | a_{ik} = 0\} \cup \{a_{ik}a_j + (-a_{jk})a_i | a_{ik} > 0, a_{jk} < 0\}$ i.e. we take the rows of A in which $a_{ik} = 0 \Leftrightarrow x_k$ does not show up ($x_1 \leq 4$ in our example) together with pairs of lower and upper bounds:

$$\begin{aligned} & \langle a_i, x\rangle \leq 0 \\ \Leftrightarrow\ & a_{ik}x_k \leq \langle -a_i, x\rangle - a_{ik}x_k \\ \Leftrightarrow\ & x_k \leq \frac{1}{a_{ik}}(\langle -a_i, x\rangle - a_{ik}x_k) \end{aligned}$$

yields an upper bound if $a_{ik} > 0$,

$$\begin{aligned} & \langle a_j, x\rangle \leq 0 \\ \Leftrightarrow\ & a_{jk}x_k \leq \langle -a_j, x\rangle - a_{jk}x_k \\ \Leftrightarrow\ & x_k \geq \frac{1}{a_{jk}}(\langle -a_j, x\rangle - a_{jk}x_k) \end{aligned}$$

yields an lower bound if $a_{jk} < 0$. The combination of lower and upper bound is

$$\frac{1}{a_{jk}}(\langle -a_j, x\rangle - a_{jk}x_k) \leq x_k \leq \frac{1}{a_{ik}}(\langle -a_i, x\rangle - a_{ik}x_k)$$

Such x_k exists if and only if

$$\begin{aligned} & \frac{1}{a_{jk}}(\langle -a_j, x\rangle - a_{jk}x_k) \leq \frac{1}{a_{ik}}(\langle -a_i, x\rangle - a_{ik}x_k) \\ \Leftrightarrow\ & \frac{1}{a_{jk}}\langle -a_j, x\rangle \leq \frac{1}{a_{ik}}\langle -a_i, x\rangle \\ \Leftrightarrow\ & a_{ik}\langle -a_j, x\rangle \geq a_{jk}\langle -a_i, x\rangle \\ \Leftrightarrow\ & 0 \geq \langle -a_{jk}a_i + a_{ik}a_j, x\rangle \end{aligned}$$

3.24 Lemma. *Let* $\sigma = \{x|Ax \leq 0\}$

$$\begin{aligned} \operatorname{elim}_k(\sigma) &= \{x|A^{/k}x \leq 0\} \\ \operatorname{proj}_k(\sigma) &= \{x|A^{/k}x \leq 0, x_k = 0\} \end{aligned}$$

Beweis. The rows of $A^{/k}$ are positive linear combinations of the rows of A, thus the inequalities of $A^{/k}$ are satisfied for points in $\sigma \Rightarrow \sigma \subset \{x | A^{/k}x \leq 0\}$. Since the variable x_k does not appear in the system $A^{/k}x \leq 0$, $\mathrm{elim}_k(\sigma) \subset \{x | A^{/k}x \leq 0\}$
Vice versa, assume x satisfies $A^{/k}x \leq 0$ and let $x_k = 0$. We want to see that there is $y \in \mathbb{R}$ s.th. $x - ye_k \in \sigma$, then $x \in \mathrm{elim}_k(\sigma)$.
Consider $A(x - ye_k)$. We want $A(x - ye_k) \leq 0$. We have

$$A(x - ye_k) = Ax - Aye_k = Ax - y(Ae_k)$$

The i-th entry of this vector is $\langle a_i, x\rangle - ya_{ik}$. We have

$$\begin{aligned} &\langle a_i, x\rangle - ya_{ik} \leq 0 \\ \Leftrightarrow\, &\langle a_i, x\rangle \leq ya_{ik} \\ \Leftrightarrow\, &\begin{cases} \frac{1}{a_{ik}}\langle a_i, x\rangle \leq y & \text{if } a_{ik} > 0 \\ \frac{1}{a_{ik}}\langle a_i, x\rangle \leq y & \text{if } a_{ik} < 0 \end{cases} \end{aligned}$$

To find a y satisfying all these inequalities, we must have

$$\max\{\frac{1}{a_{ik}}\langle a_i, x\rangle : a_{ik} > 0\} \leq y \leq \min_j\{\frac{1}{(-a_{jk})}\langle -a_j, x\rangle : a_{jk} < 0\}$$

Such a y exists, since x satisfies $A^{/k}x \leq 0$, i.e. $\langle a_{ik}a_j + (-a_{jk}a_i, x\rangle \leq 0 \,\forall\, i, j$ with $a_{ik} > 0$, $a_{jk} < 0$, i.e. $\frac{1}{a_{ik}}\langle a_i, x\rangle \leq (\frac{1}{-a_{jk}})\langle -a_j, x\rangle \,\forall$ such i, j and thus also for their min and max. □

3.25 Lemma. *A finite intersection of affine closed halfspaces of the form* $\sigma = \{\begin{pmatrix} x \\ w \end{pmatrix} \in \mathbb{R}^n \times \mathbb{R}^m | Ax \leq w\}$ *is a cone.*

Beweis. Claim:

$$\sigma = \mathrm{Cone}(\{\pm \begin{pmatrix} e_i \\ Ae_i \end{pmatrix}, 1 \leq i \leq n\} \cup \{\begin{pmatrix} 0 \\ e_j \end{pmatrix} : 1 \leq j \leq m\},$$

" $\subset$ ": Let $\begin{pmatrix} x \\ w \end{pmatrix}$ satisfy $Ax \leq w$. Then

$$\begin{pmatrix} x \\ w \end{pmatrix} = \sum_{i=1}^{n} |x_i|(\mathrm{sgn}(x_i)\begin{pmatrix} e_i \\ Ae_i \end{pmatrix}) + \sum_{j=1}^{m}(w_j - (Ax)_j)\begin{pmatrix} 0 \\ e_j \end{pmatrix}$$

since $Ax = A(x_1e_1 + \ldots + x_ne_n) = x_1(Ae_1) + \ldots + x_n(Ae_n)$. This is a nonnegative linear combination of the generators above, since $|x_i| \geq 0$, $w_j - (Ax)_j \geq 0$ as $Ax \leq w$.
" $\supset$ ": Let $\begin{pmatrix} x \\ w \end{pmatrix}$ be in the cone, i.e.

$$\begin{pmatrix} x \\ w \end{pmatrix} = \sum_{i=1}^{n} \lambda_i \begin{pmatrix} e_i \\ Ae_i \end{pmatrix} + \sum_{i=1}^{n} \mu_i \begin{pmatrix} -e_i \\ -Ae_i \end{pmatrix} + \sum_{j=1}^{m} v_j \begin{pmatrix} 0 \\ e_j \end{pmatrix}$$

with $\lambda_i, \mu_i, v_i \geq 0$, then $x_i = \lambda_i - \mu_i$,

$$\begin{aligned} &w_j = (Ax)_j + v_j, \text{ as } v_j \geq 0 \\ &w_j - (Ax)_j = v_j \geq 0 \,\forall\, j \Rightarrow w - Ax = v \geq 0 \Rightarrow w - Ax \geq 0 \\ &w_j \leq w_j - v_j = (Ax)_j \Rightarrow Ax \leq w \end{aligned}$$

□

An intersection of finitely many halfspaces $\{x|Ax \leq 0\}$ can be written as

$$\{\begin{pmatrix} x \\ w \end{pmatrix} |Ax \leq w\} \cap \{w_1 = \ldots = w_m = 0\}$$

Thus, we now set variables equal to 0 and make sure that the property of being a cone is maintained.

3.26 Lemma. *Let $\sigma = \mathrm{Cone}(S)$ be a cone, then also $\sigma \cap \{x_k = 0\}$ is a cone.*

Beweis. Let $S = \{y_1, \ldots, y_r\}$. We claim

$$\sigma \cap \{x_k = 0\} = \mathrm{Cone}(\{y_i : (y_i)_k = 0\} \cup \{y_{ik}y_j + (-y_{jk})y_i | y_{ik} > 0, y_{jk} < 0\})$$

" $\supset$ ": These generators all have coordinate k equal to 0. Furthermore, they are positive linear combinations of vectors in S.
" $\subset$ ": Let $v = t_1 y_1 + \ldots + t_r y_r \in \sigma$, $t_i \geq 0$, $v_k = 0 \Rightarrow \sum t_i y_{ik} = 0$.
If $t_i y_{ik} = 0 \,\forall\, i \Rightarrow$ for one summand, $t_i y_{ik}$, either $t_i = 0$, but then the summand can be left out, or $y_{ik} = 0 \Rightarrow v \in \mathrm{Cone}(y_i | y_{ik} = 0)$. Else we sort in positive and negative summands:

$$\sum_{i:y_{ik}>0} t_i y_{ik} = \sum_{j:y_{jk}<0} t_j(-y_{jk}) =: R > 0$$

Then we write v as

$$\begin{aligned}
v &= \sum t_i y_i \\
&= \sum_{i:y_{ik}=0} t_i y_i + \sum_{i:y_{ik}>0} t_i y_i + \sum_{j:y_{jk}<0} t_j y_j \\
&= \sum_{i:y_{ik}=0} t_i y_i + \frac{1}{R} \sum_{i:y_{ik}>0} R t_i y_i + \frac{1}{R} \sum_{j:y_{jk}<0} R t_j y_j \\
&= \sum_{i:y_{ik}=0} t_i y_i + \frac{1}{R} \sum_{i:y_{ik}>0} \Big(\sum_{j:y_{jk}<0} t_j(-y_{jk})\Big) t_i y_i + \frac{1}{R} \sum_{j:y_{jk}<0} \Big(\sum_{i:y_{ik}>0} t_i y_{ik}\Big) t_j y_j \\
&= \sum_{i:y_{ik}=0} t_i y_i + \frac{1}{R} \sum_{\substack{i:y_{ik}>0 \\ j:y_{jk}<0}} t_i t_j((-y_{jk})y_i + y_{ik}y_j)
\end{aligned}$$

$\Rightarrow$ v is in the cone as required. $\square$

Now we finally prove part 2) of Theorem 3.21.
<u>Cones:</u> $\sigma \in \mathbb{R}^n$ is a cone $\Leftrightarrow$ σ is the intersection of a linear subspace with finitely many closed halfspaces, i.e. inside span(σ) we can write $\sigma = H^+_{m_1} \cap \ldots \cap H^+_{m_r}$.

Beweis. " $\Rightarrow$ ": Let σ ba a cone,

$$\begin{aligned}
\sigma &= \mathrm{Cone}(S) = \mathrm{Cone}(y_1, \ldots, y_r) \\
&= \{t_1 y_1 + \ldots + t_r y_r,\ t_i \geq 0\} \\
&= \Pi_{n+1} \circ \ldots \circ \Pi_{n+r} \underbrace{(\{\begin{pmatrix} x \\ t \end{pmatrix} \in \mathbb{R}^{n \times r} | Yt = x,\ t_i \geq 0\})}_{\text{intersection of finitely many halfspaces } (Yt \leq x, Yt \geq x, t \geq 0)}
\end{aligned}$$

where Y is the matrix with the y_i as columns.
The right hand side is a finite intersection of halfspaces (resp. can be written as such via $Yt - x \leq 0$, $-(Yt - x) \leq 0$, $(-t_i) \leq 0$). Lemma 3.24 shows that for such σ' of the form $\sigma' = \{x | Ax \leq 0\}$ also elim_k and proj_k are finite intersections of halfspaces (of the form $\{x | A^{/k}x \leq 0\}$). Hence our σ is a finite intersection of halfspaces.

"$\Leftarrow$": Assume σ is an intersection of finitely many halfspaces (we can assume that by using two versions ≤ 0 and ≥ 0 for our linear equalities $= 0$ for the linear subspace), i.e. σ is of the form $\sigma = \{x | Ax \leq 0\} \Rightarrow \tilde{\sigma} = \{\begin{pmatrix} x \\ w \end{pmatrix} | Ax \leq w\}$ is a cone by lemma 3.24, and iteratively applying lemma 3.25 we obtain $\sigma = \tilde{\sigma} \cap \{w_1 = 0\} \cap \ldots \cap \{w_r = 0\} = \tilde{\sigma} \cap \{w_1 = \ldots = w_r = 0\}$ is a cone. $\square$

We have finally completed the proof of the duality theorem 3.21 that states that cones (polytopes, polyhedra) defined via generators are the same as cones (polytopes, polyhedra) defined via inequalities.
Why is this so good to know?

- for the theory: to finally prove that $\sigma^\vee$ is a cone, will be done soon now
- in practice: assume you are given x and want to check whether $x \in P$? That's easy if you have the inequalities for P.
  Assume you have the task to find a point $x \in P$: that's easy if you have P given by generators!
- Computations with polytopes, cones, ...: POLYMAKE (TU Berlin) OSCAR

3.27 Proposition. *$\sigma^\vee$ is a cone.*

Beweis. Because of theorem 3.21, we have $\sigma = H^+_{m_1} \cap \ldots \cap H^+_{m_r}$. Let $\sigma' \subset (\mathbb{R}^n)^\vee$, $\sigma' = \text{Cone}(m_1, \ldots, m_r)$.
Claim: $\sigma' = \sigma^\vee$, in particular, $\sigma^\vee$ is a cone.
"$\subset$": Let $m \in \sigma' \Rightarrow m = \sum \lambda_i m_i$, $\lambda_i \geq 0$. Since $\langle m_i, u\rangle \geq 0 \,\forall\, u \in \sigma$, $\forall\, i$ ($\sigma \subset H^+_{m_i}$), also $\langle m, u\rangle \geq 0 \,\forall\, u \in \sigma \Rightarrow m \in \sigma^\vee$.
"$\subset$": Assume not, then $\sigma' \subsetneq \sigma^\vee$, let $m \in \sigma^\vee \setminus \sigma'$. Then there exists a seperating hyperplane $u \in (\sigma')^\vee$ with $\langle m, u\rangle < 0$. As σ' is generated by the m_i and $u \in (\sigma')^\vee$
$\Rightarrow \langle m_i, u\rangle \geq 0 \,\forall\, i \Rightarrow u \in H^+_{m_i} \,\forall\, i \Rightarrow u \in \sigma = (\sigma^\vee)^\vee \Rightarrow \langle m, u\rangle \geq 0$ ↯. $\square$

3.28 Remark. Generators of the dual cone are normal vectors to the hyperplanes which cut the cone out, see Figure 21.

What if σ is contained in a linear subspace? (Figure 229

We use a pair of two inequalities, e.g. $x_3 \leq 0$, $x_3 \geq 0$. We thus obtain two normal vectors $\pm e_3$ whose non-negative combination generate a subspace, namely the annihilator of the subspace generated by σ in which σ lives.
The dual cone is pictured in Figure 23

[20]Image from Hannah Markwig.
[21]Image from Hannah Markwig.
[22]Image from Hannah Markwig.

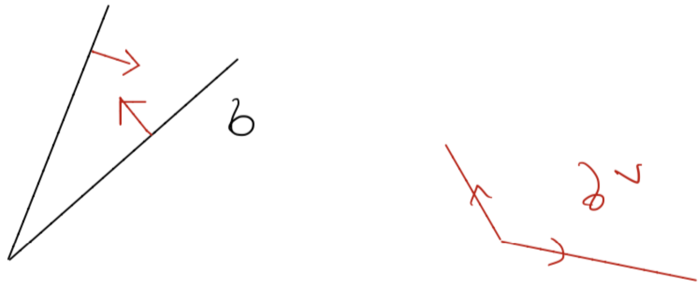


Abbildung 21: σ and $\sigma^\vee$ [20]

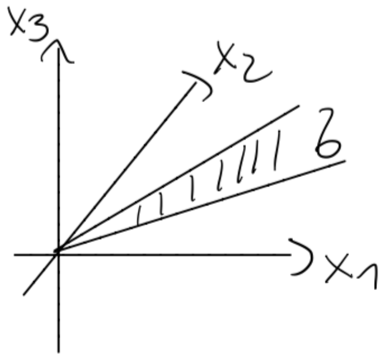


Abbildung 22: σ in a linear subspace [21]

Abbildung 23: the dual cone [22]

4 Faces of cones and polytopes

Remember the supporting hyperplanes $H_m = \{u | \langle m, u\rangle = 0\}$ for $m \in \sigma^\vee \setminus \{0\}$.
We now also use $H_0 = \mathbb{R}^n$ and define:

4.1 Definition. A **face** of a cone σ is $\tau = \sigma \cap H_m$ for some $m \in \sigma^\vee$. For $m = 0$, σ is a face of itself.

4.2 Example. $\sigma = \operatorname{Cone}\left(\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}\right)$

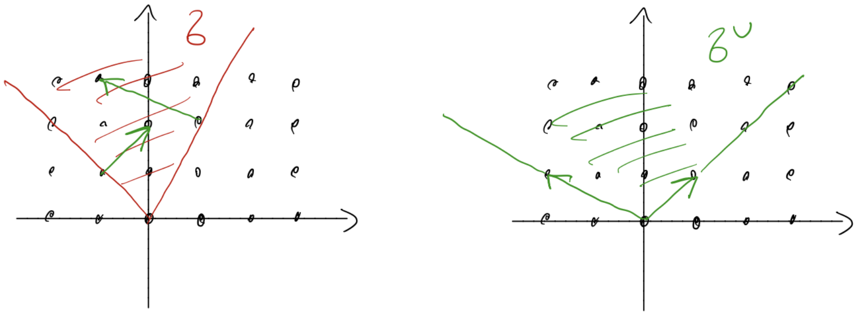


Abbildung 24: σ and $\sigma^\vee$ [23]

$\sigma^\vee = \operatorname{Cone}\left(\begin{pmatrix} -2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right)$. Faces of σ:

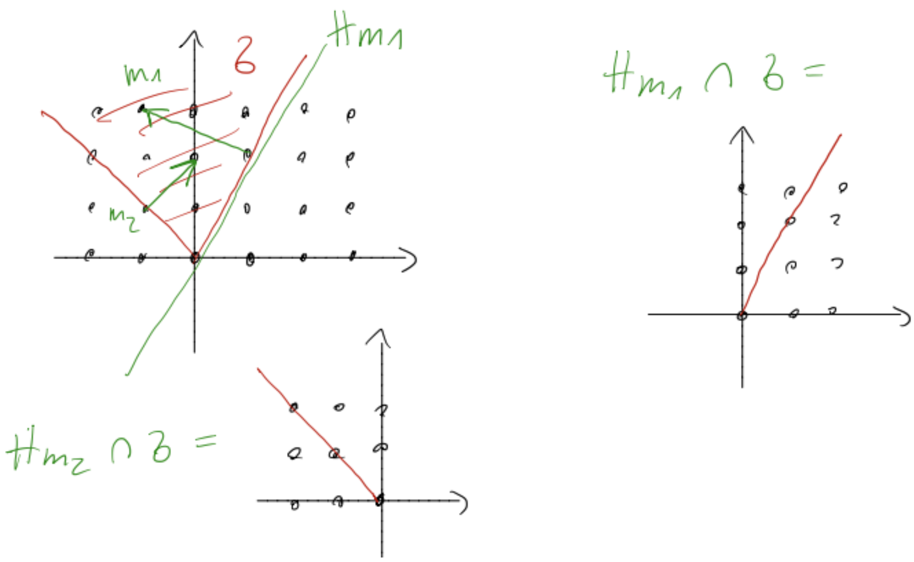


Abbildung 25: Faces of σ [24]

If we take $m = \lambda_1 m_1 - \lambda_2 m_2$, $\lambda_i > 0$,

[23]Image from Hannah Markwig.
[24]Image from Hannah Markwig.
[25]Image from Hannah Markwig.

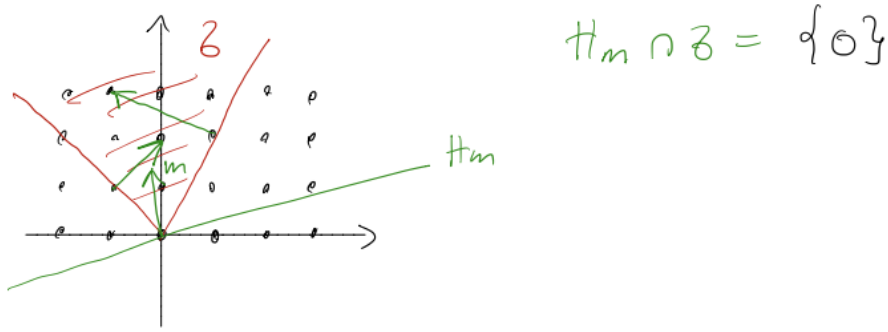


Abbildung 26: $H_m \cap \sigma$ [25]

for $m = 0$, we obtain σ. Altogether, we have 4 faces: the cone point (origin), two rays and σ itself.

4.3 Example. Faces of σ: origin, 4 rays, 4 2-dimensional faces (the intersection with the defining hyperplane, i.e. the hyperplanes whose inequalities cut out σ), σ itself.

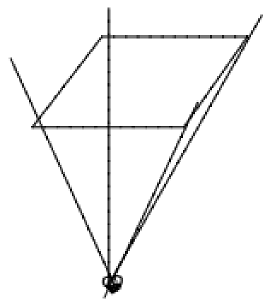


Abbildung 27: σ [26]

4.4 Lemma. *Let σ be a cone.*

(1) σ has finitely many faces.

(2) A face of σ is a cone itself.

(3) The intersection of two faces of σ is a face of σ.

(4) The face τ' of a face τ of σ is a face of σ.

Beweis. Assume $\sigma = \mathrm{Cone}(S)$. $H_m \cap \sigma$ is generated by all $u \in S$ s.th. $\langle m, u \rangle = 0$. ($S = \{u_1, \ldots, u_k\}, u \in \sigma, u = \sum \lambda_i u_i, \lambda_i \geq 0, u \in H_m \Leftrightarrow \langle m, u \rangle = 0$ $\Leftrightarrow \sum \lambda_i \langle m, u_i \rangle = 0 \Leftrightarrow u$ is a non-negligible combination of all u_i with $\langle m, u_i \rangle = 0$.) This proves 2). In particular, there are only finitely many faces, as there are only finitely many

[26]Image from Hannah Markwig.

subsets of the finite set S. This proves 1).
3) Let $\tau_1 = H_{m_1} \cap \sigma, \tau_2 = H_{m_2} \cap \sigma$. Since $m_i \in \sigma^\vee$, $\langle m_i, u\rangle \geq 0 \,\forall\, u \in \sigma$.

$$\begin{aligned} H_{m_1+m_2} \cap \sigma &= \{u \in \sigma | \langle m_1 + m_2, u\rangle = 0\} \\ &= \{u \in \sigma | \underbrace{\langle m_1, u\rangle}_{\geq 0} + \underbrace{\langle m_2, u\rangle}_{\geq 0} = 0\} \\ &= \{u \in \sigma | \langle m_1, u\rangle = 0 \text{ and } \langle m_2, u\rangle = 0\} \\ &= \{u \in \sigma | \langle m_1, u\rangle = 0\} \cap \{u \in \sigma | \langle m_2, u\rangle = 0\} \\ &= (H_{m_1} \cap \sigma) \cap (H_{m_2} \cap \sigma) = \tau_1 \cap \tau_2 \end{aligned}$$

4) Let $\tau = \sigma \cap H_{m_1}, \gamma = \tau \cap H_{m_2}, m_1 \in \sigma^\vee, m_2 \in \tau^\vee$. Assume $\sigma = \mathrm{Cone}(u_1, \ldots, u_k, u_{k+1}, \ldots, u_r)$ s.th.

$$\begin{aligned} \langle m_1, u_i\rangle &= 0 \text{ for } i = 1, \ldots, k \\ \text{and } \langle m_1, u_i\rangle &> 0 \text{ for } i = k+1, \ldots, r \end{aligned}$$

Then $u_i \in H_{m_1} \cap \sigma = \tau$ for $i = 1, \ldots, k \Rightarrow \langle m_2, u_i\rangle \geq 0 \,\forall\, i = 1, \ldots, k$, as $m_2 \in \tau^\vee$.
Pick p big enough, $p >> 0$, s.th.

$$p\langle m_1, u_i\rangle + \langle m_2, u_i\rangle \geq 0 \quad \forall\, i = k+1, \ldots, r$$

This is possible, since $\langle m_1, u_i\rangle > 0$. (Note that $m_2 \in \tau^\vee$ but not necessarily in $\sigma^\vee$, so that $\langle m_2, u_i\rangle < 0$ is possible.)
Then $pm_1 + m_2 \in \sigma^\vee$. Let $u \in \sigma, u = \sum \lambda_i u_i, \lambda_i \geq 0$.

$$\begin{aligned} \langle pm_1 + m_2, u\rangle &= \sum \lambda_i \langle pm_1 + m_2, u_i\rangle \\ &= \sum_{i=1}^{k} \underbrace{\lambda_i}_{\geq 0} \underbrace{\langle m_2, u_i\rangle}_{\geq 0} + \sum_{i=k+1}^{r} \underbrace{\lambda_i}_{\geq 0} \underbrace{(p\langle m_1, u_i\rangle + \langle m_2, u_i\rangle)}_{\geq 0} \geq 0. \end{aligned}$$

$$\begin{aligned} \sigma \cap H_{pm_1+m_2} &= \{u \in \sigma | \langle pm_1 + m_2, u\rangle = 0\} \\ &= \{u \in \sigma | p\langle m_1, u\rangle + \langle m_2, u\rangle = 0\} \end{aligned}$$

If $\langle m_1, u\rangle > 0$, then $p\langle m_1, u\rangle >> 0 \Rightarrow p\langle m_1, u\rangle + \langle m_2, u\rangle \neq 0 \Rightarrow \langle m_1, u\rangle = 0$
$\Rightarrow u \in H_{m_1} \cap \sigma = \tau$ but then also $\langle m_2, u\rangle = 0$ and $u \in \tau \cap H_{m_2} = \gamma$.
Vice versa, every $u \in \gamma$ is in $H_{pm_1+m_2} \cap \sigma$. $\square$

4.5 Example. $\sigma = \mathrm{Cone}\left(\begin{pmatrix} -3 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right)$, $\sigma^\vee = \mathrm{Cone}\left(\begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \end{pmatrix}\right)$, shown in Figure 28.

$m_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$, $H_{m_1} \cap \sigma = \tau$, $\tau^\vee = \mathrm{Cone}\left(\begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right)$.
Let $m_2 = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \in \tau^\vee$, $H_{m_2} \cap \tau = \{0\}$.
Since here, $m_2 \in \tau^\vee \setminus \sigma^\vee, \exists\, u \in \sigma : \langle m_2, u\rangle < 0$,
e.g. $u = \begin{pmatrix} -3 \\ 1 \end{pmatrix} : \left\langle \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \begin{pmatrix} -3 \\ 1 \end{pmatrix} \right\rangle = -7$.
But if we add pm_1, we get back to $\sigma^\vee$. Here, $p = 2$ works.

[27] Image from Hannah Markwig.

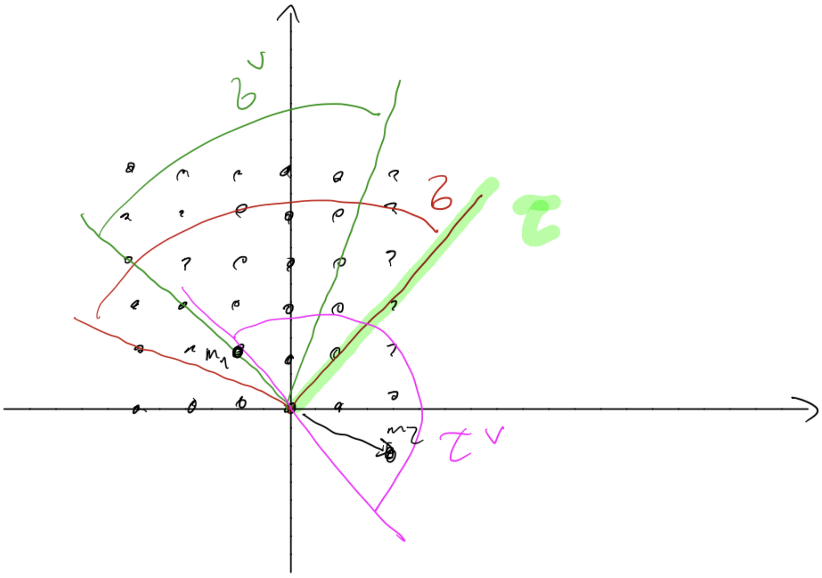


Abbildung 28: $\sigma, \sigma^\vee, \tau, \tau^\vee$ [27]

<u>Notation:</u> A **facet** is s face of codimension 1, (i.e. dimension one less than σ). An **edge** is a face of dimension 1. A **strict face** is a face which is not σ.

4.6 Lemma. *Every strict face is contained in a facet.*

Beweis. It is enough to show that every $\tau = H_m \cap \sigma$ of codimension > 1 is contained in a face of strictly bigger dimension. Let $V = span(\sigma), W = span(\tau), \sigma = \mathrm{Cone}(u_1, \dots, u_r)$. The equivalence classes $\overline{u_i} \in V/W$ are contained in $H^+_{\overline{m}}$. We can move this halfspace around s.th. it still contains all $\overline{u_i}$, but such that one $\overline{u_i} \neq 0$ moves into $H_{\overline{\tilde{m}}}$:

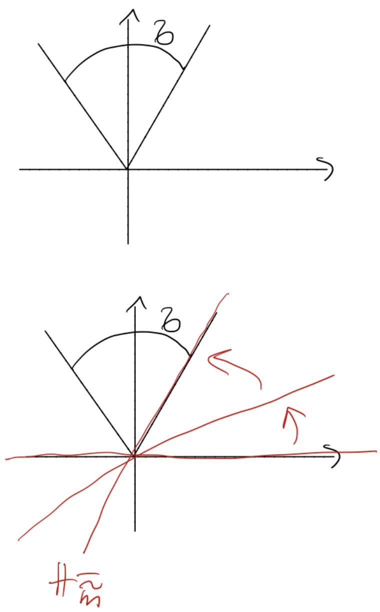


Abbildung 29: $\sigma, H_{\overline{\tilde{m}}}$ [28]

$\tau =$ origin, $m = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $H_m^+ =$ upper halfplane. Turn such that one edge moves into the hyperplane (see Figure 29). Then $H_{\tilde{m}} \cap \sigma$ is bigger than τ, since it additionally contains this u_i. $\square$

4.7 Remark. If codim $\tau = 2$ (as in Figure 29), then V/W is 2-dimensional and there are exactly 2 such faces/ choices of hyperplanes. It follows:
If τ is a face of codimension 2, then τ is contained in precisely 2 facets F_1, F_2, and $\tau = F_1 \cap F_2$. By induction we can conclude: Every strict face is the intersection of all facets that contain it,

$$\tau = \bigcap_{\tau \subset F_i,\, F_i \text{ facet}} F_i.$$

This is because, if τ is of codim > 2, we find a facet γ with $\tau \subset \gamma$. By induction, τ is the intersection of all facets of γ that contain it. But a facet of γ is a face of σ of codimension 2 and can be written as the intersection of two facets as above.

4.8 Lemma. *The topological boundary of a cone is equal to the union of its strict faces (or facets).*

Beweis. Let $p \in \tau = H_m \cap \sigma$ with $m \in \sigma^\vee \setminus \{\sigma\}$. As $\sigma \subset H_m^+$ and $\langle m, p\rangle = 0$ there exist points q arbitrarily close to p which are not contained in $\sigma \Rightarrow p$ is contained in the topological boundary. Vice versa, let p be a point in the topological boundary. Let $p_i \to p$ be a converging sequence of points $p_i \notin \sigma$.
For each p_i, there exists a separating hyperplane H_{m_i}, $m_i \in \sigma^\vee$ s.th. $\langle m_i, p_i\rangle < 0$. We can choose m_i s.th. $||m_i|| = 1$, then the sequence $(m_i)_{i\in I}$ is bounded and thus has a converging partial sequence. Without restriction $m_i \to m$, $m \in \sigma^\vee \setminus \{0\}$ (as $m_i \in \sigma^\vee$ and $\sigma^\vee$ is closed) $m \neq 0$ as $||m_i|| = 1 \,\forall\, i$. As $p \in \sigma$, $\langle m, p\rangle \geq 0$. But

$$\langle m, p\rangle = \lim \underbrace{\langle m_i, p_i\rangle}_{<0}$$
$$\Rightarrow \langle m, p\rangle = 0 \Rightarrow p \in H_m \cap \sigma \Rightarrow p \text{ is contained in a strict face of } \sigma.$$

As every strict face is the intersection of the facets that contain it, we can as well take the union of all facets. $\square$

4.9 Remark. If $\operatorname{span}(\sigma) = \mathbb{R}^n$ and τ is a facet, then $\exists$ normal vector $m_\tau \in \sigma^\vee$ which is unique up to multiple with a scalar, s.th. $H_{m_\tau} \cap \sigma = \tau$. This is true since $\operatorname{span}(\tau)$ is a hyperplane, whose normal vector is unique up to multiple with a scalar.
From Chapter 3, we know already:

$$\sigma = \bigcap_{\tau \text{ facet}} H_{m_\tau}^+.$$

Now, we can phrase this differently and reprove the equality (without the constructive approach from Chapter 3):

[28] Image from Hannah Markwig.

Beweis. "$\subset$": since $m_\tau \in \sigma^\vee$, $\Rightarrow \sigma \subset H^+_{m_\tau}\ \forall \tau$.
"$\supset$": Assume there exists $p \in \bigcap H^+_{m_\tau}$, $p \notin \sigma$. Let q be a point in the topological interior of σ (σ°). Let p' be the last point in the line $\overline{pq}$ which is still in σ. Then p' is in the topological boundary of σ and thus contained in a facet τ. But then $\langle m_\tau, q\rangle > 0$ (as q is in the interior of σ, so not in the topological boundary, so not in the facet τ), $\langle m_\tau, p'\rangle = 0$ $\Rightarrow \langle m_\tau, p\rangle < 0$ ↯ $p \in H^+_{m_\tau}$ □

4.10 Definition. A cone σ is called **rational** if $\sigma = \mathrm{Cone}(S)$ for a finite subset $S \subset \mathbb{Z}^n$.

4.11 Remark. The dual cone of a rational cone, and faces of a rational cone, are rational themselves, since the normal vectors of facets are again rational.

4.12 Definition. A cone is called **strictly convex** if $\{0\}$ is a face.

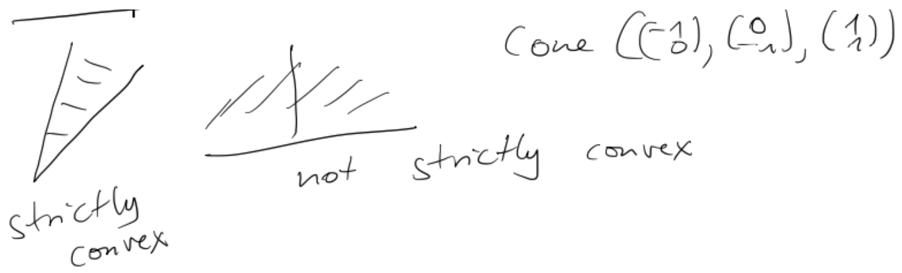


Abbildung 30: (strictly) convex cones [29]

4.13 Example. For strictly convex rational cones, there is a nice generating set: Let ρ be an edge of σ. As σ is strictly convex, ρ is a ray (a half line). As σ is rational, there must be integer points on ρ. Denote by u_ρ teh first integer point we reach on ρ starting from 0 and call it the **ray generator** of ρ.

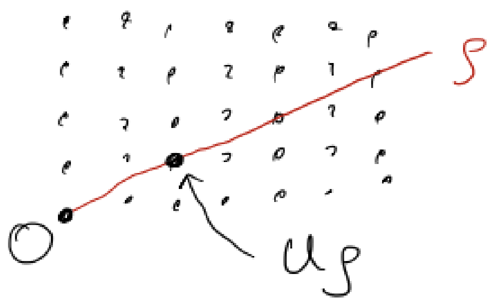


Abbildung 31: ρ and the ray generator u_ρ [30]

4.14 Lemma. *If σ is strictly convex and H^+ a halfspace with $H^+ \cap \sigma \neq \{0\}$, then H^+ contains a ray of σ, see figure 32.*

Beweis. Induction on $\dim \sigma$.
If $\dim \sigma = 1$ ($\sigma = \mathrm{Cone}(u)$), the statement is clear, since H^+ contains λu, $\lambda > 0$, if it contains $u \neq 0$, $u \in \sigma$, hence the whole ray.
If $\dim \sigma = n$: since σ is strictly convex, $H^+ \not\subset \sigma$, hence in H^+ there are as well points $p \neq 0$ in σ as points $q \neq 0$ which are not in σ.
If $\sigma \subset H$ choose $q \notin \sigma$, $q \notin H$, see figure 33.

[29] Image from Hannah Markwig.
[30] Image from Hannah Markwig.
[31] Image from Hannah Markwig.

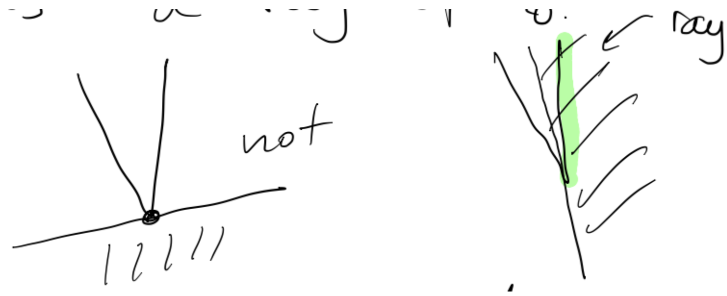


Abbildung 32: (Not a) ray of σ [31]

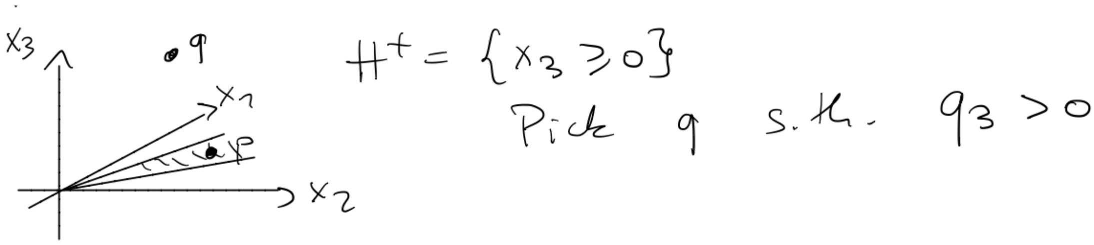


Abbildung 33: $\sigma \subset H$ [32]

If $\sigma \not\subset H$ pick $p \notin H$, see figure 34.

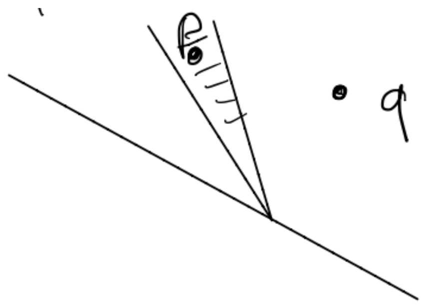


Abbildung 34: $\sigma \not\subset H$ [33]

In any case, the line $\overline{pq}$ does not pass through 0. On the line $\overline{pq}$ there is a point p' in the topological boundary of σ. This point, p', must be contained in a strict face τ of σ, $\dim \tau < n$.
We have $\tau \cap H^+ \neq \{0\}$, since $p' \in \tau \cap H^+$.
By induction, a ray of τ is contained in H^+. But then also a ray of σ is contained in H^+. □

4.15 Lemma. *A strictly convex rational cone σ is generated by its ray generators.*

Beweis. Let σ be the cone and $\sigma' = \mathrm{Cone}(\text{ray generators})$. Then $\sigma' \subset \sigma$. Assume $\exists\, p \in \sigma \backslash \sigma'$. Then there exists a seperating hyperplane $H_m, m \in (\sigma')^\vee : \langle m, p\rangle < 0$. As $m \in (\sigma')^\vee$ we have $\langle m, u\rangle \geq 0 \,\forall\, u \in \sigma' \;\Rightarrow\; H_m^- \cap \{\sigma'\} = \{0\}$. But H_m^- containes p, thus $H_m^- \cap \sigma \neq \{0\}$. By lemma 4.15, H_m^- contains a ray of σ ↯ as all rays are in σ'. □

[32]Image from Hannah Markwig.
[33]Image from Hannah Markwig.

Now we turn to faces of polytopes. Here, we need affine supporting hyperplanes:

$$H_{m,b} = \{u \in \mathbb{R}^n | \langle m, u \rangle = b\}$$
$$H_{m,b}^+ = \{u \in \mathbb{R}^n | \langle m, u \rangle \geq b\}$$

4.16 Definition. A **face** of a polytope P is

$$Q = H_{m,b} \cap P \quad \text{for} \quad m \in (\mathbb{R}^n)^\vee, b \in \mathbb{R}$$

satisfying $P \subset H_{m,b}^+$. P is a face of itself. $\emptyset$ is a face of P.

4.17 Example. Faces of the triangle in figure 35 are: $\emptyset$, the 3 vertices, the 3 edges and the triangle itself.

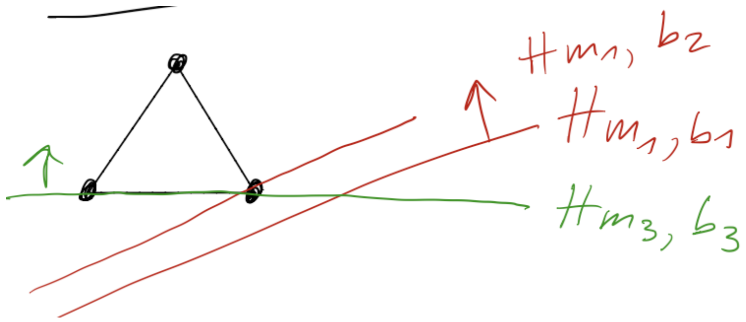


Abbildung 35: Supporting hyperplanes which cut out nothing (b_1), a vertex (b_2) or an edge (b_3). [34]

4.18 Remark. Let P be a polytope, consider it inside the affine space $\subset \{x_{n+1} = 1\} \subset \mathbb{R}^{n+1}$, and let σ_P be the cone over P, (Cone(P)).

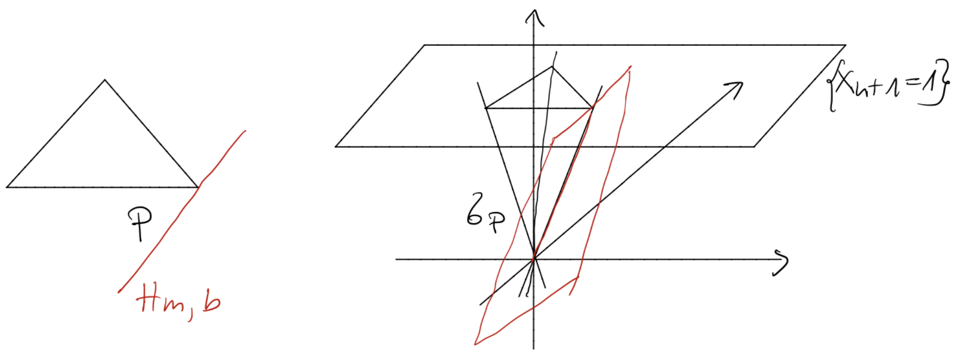


Abbildung 36: P and σ_P [35]

Faces of σ_P are precisely the cones over faces of P (where $\{0\}$ is the cone of $\emptyset$, the empty face of P):
Let τ be a face of $\sigma_P \Rightarrow \tau = H_m \cap \sigma_P$ for $m \in \sigma_P^\vee$. Thus $\sigma_P \subset H_m^+ \Rightarrow P \subset (H_m^+ \cap (\mathbb{R}^n \times \{1\}))$, and $H_m \cap (\mathbb{R}^n \times \{1\})$ is an affine hyperplane cutting out a face Q s.th. $\tau = \sigma_Q$.
Vice versa, let Q be a face of P, then $Q = P \cap H_{m,b}$ for some $P \subset H_{m,b}^+$. $H_{m,b} \times \{1\}$ together with 0 generates a hyperplane $H_{\tilde{m}}$ in $\mathbb{R}^{n+1}$ s.th. $\sigma \subset H_{\tilde{m}}^+$ and $\sigma \cap H_{\tilde{m}}$ is a face τ s.th. $\tau = \sigma_Q$.
Accordingly the following results on faces of cones hold for polytopes analogously:

[34] Image from Hannah Markwig.
[35] Image from Hannah Markwig.

4.19 Lemma. *Let P be a polytope.*

(1) P has finitely many faces.

(2) A face is a polytope.

(3) The intersection of two faces is a face.

(4) The face of a face of P is a face of P.

(5) Every strict face is the intersection of all facets containing it.

(6) The topological boundary of P is equal to the union of its strict faces (or facets).

(7) The defining inequalities of a polytope P (of dimension n in $\mathbb{R}^n$) are given by the normal vectors of its facets.

4.20 Definition. A **lattice polytope** is the convex hull of a finite set $S \subset \mathbb{Z}^n$.
The cone over a lattice polytope is rational and hence generated by its ray generators.

4.21 Corollary. *A lattice polytope is the convex hull of its vertices (**vertex** = face of* $\dim 0$*).*

Beweis. This follows since σ_P is generated by its rays and the rays are the cones over the vertices. □

4.22 Remark. In fact, more generally: A polygon is the convex hull of its vertices.
This holds true, since the proof that a rational cone is generated by its ray generators can be modified in such a way that we see: a cone is generated by a set u_ρ, where u_ρ is an arbitrary point $\neq 0$ on ρ.
Rational was only needed to define u_ρ as ray generator.

4.23 Proposition. *Let $P = \mathrm{Conv}(K)$, then K contains the vertices of P. In particular, the set of vertices is the unique minimal generating set of P.*

Beweis. Let σ be the cone of P. Assume a vertex Q of P was not in K. Let τ be the cone over Q, then τ is a ray of σ. Q must be equal to a convex combination of K:
$Q = \lambda_1 u_1 + \ldots + \lambda_m u_m, \lambda_i \geq 0, \sum \lambda_i = 1$ for $K = \{u_1, \ldots, u_m\}$. Divide this sum into two summands which are not 0 (this is possible, since we assume $Q \notin K$). $Q = v + w$. Then

$$Q \times \{1\} = v \times \{1\} + w \times \{1\}, \quad \text{and} \quad v \times \{1\}, w \times \{1\} \in \sigma,$$

but $v \times \{1\}, w \times \{1\} \notin \tau$, since τ only contains positive multiples of $Q \times \{1\}$:
If $\lambda \cdot (Q, 1) = (v, 1) \Rightarrow \lambda = 1 \Rightarrow v = Q$ ↯.
Homework: If $v, w \in \sigma, v + w \in \tau \Rightarrow v, w \in \tau$ ↯. □

Also an analogue of the dual cone $\sigma^\vee$ exists for polytopes:

4.24 Definition. Let $P \in \mathbb{R}^n$ be a polytope. The **polar set** $P^\triangle$ is defined by

$$P^\triangle := \{m \in (\mathbb{R}^n)^\vee | \langle m, u \rangle \leq 1 \, \forall \, u \in P\} \subset (\mathbb{R}^n)^\vee$$

Abbildung 37: P and $P^{\triangle}$ [36]

4.25 Example. $P = \operatorname{Conv}\left(\begin{pmatrix} -1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right)$,
then $P^{\triangle} = \operatorname{Conv}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \end{pmatrix}\right)$, see figure 37

Since $\left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\rangle \leq 1, \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\rangle \leq 1, \ldots$ all 16 combinations ≤ 1, also for convex combinations. $\left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \lambda_1 \begin{pmatrix} -1 \\ -1 \end{pmatrix} + \lambda_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} + \lambda_3 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \lambda_4 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\rangle =$
$\lambda_1 \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \end{pmatrix} \right\rangle + \ldots \leq 1$, since every scalar product is ≤ 1 and $\sum \lambda_i = 1$.

4.26 Lemma. *Let σ_P be the cone over P, then*

$$-P^{\triangle} = \sigma_P^{\vee} \cap ((\mathbb{R}^n)^{\vee} \times \{1\})$$

Beweis. "$\supset$": Let $(m, 1) \in \sigma_P^{\vee} \cap ((\mathbb{R}^n)^{\vee} \times \{1\}) \Rightarrow (m, 1) \in \sigma_P^{\vee}$.
If $P = \operatorname{Conv}(u_1, \ldots, u_r) \Rightarrow \sigma_P = \operatorname{Cone}\left(\begin{pmatrix} u_1 \\ 1 \end{pmatrix}, \ldots, \begin{pmatrix} u_r \\ 1 \end{pmatrix}\right)$. As

$$\begin{aligned} \begin{pmatrix} m \\ 1 \end{pmatrix} \in \sigma_P^{\vee} \Rightarrow & \left\langle \begin{pmatrix} m \\ 1 \end{pmatrix}, \begin{pmatrix} u_i \\ 1 \end{pmatrix} \right\rangle \geq 0 \,\forall\, i \\ & \Leftrightarrow \langle m, u_i \rangle + 1 \geq 0 \,\forall\, i \\ & \Leftrightarrow \langle m, u_i \rangle \geq -1 \,\forall\, i \\ & \Leftrightarrow \langle -m, u_i \rangle \leq 1 \,\forall\, i \end{aligned}$$

Let $u \in P$, then $u = \lambda_1 u_1 + \ldots + \lambda_r u_r$ for $\lambda_i \geq 0, \sum \lambda_i = 1$

$\Rightarrow \langle -m, u \rangle = \langle -m, \sum \lambda_i u_i \rangle = \sum \lambda_i \langle -m, u_i \rangle \leq \sum \lambda_i = 1 \quad \Rightarrow -m \in P^{\triangle} \Rightarrow m \in -P^{\triangle}$

"$\subset$": Let $-m \in P^{\triangle} \Rightarrow \langle -m, u_i \rangle \leq 1 \,\forall\, i \Rightarrow \langle m, u_i \rangle \geq -1 \Rightarrow \langle m, u_i \rangle + 1 \geq 0$.

[36] Image from Hannah Markwig.

Let $\begin{pmatrix} u \\ l \end{pmatrix} \in \sigma_P$

$$\begin{aligned}
&\Rightarrow \begin{pmatrix} u \\ l \end{pmatrix} = \lambda_1 \begin{pmatrix} u_1 \\ 1 \end{pmatrix} + \ldots + \lambda_r \begin{pmatrix} u_r \\ 1 \end{pmatrix},\ \lambda_i \geq 0 \\
&\Rightarrow \left\langle \begin{pmatrix} m \\ 1 \end{pmatrix}, \begin{pmatrix} u \\ l \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} m \\ 1 \end{pmatrix}, \lambda_1 \begin{pmatrix} u_1 \\ 1 \end{pmatrix} + \ldots + \lambda_r \begin{pmatrix} u_r \\ 1 \end{pmatrix} \right\rangle \\
&= \lambda_1 \underbrace{\left\langle \begin{pmatrix} m \\ 1 \end{pmatrix}, \begin{pmatrix} u_1 \\ 1 \end{pmatrix} \right\rangle}_{=\langle m,u_1\rangle+1\geq 0,\text{ as } \langle -m,u_i\rangle\leq 1, \langle m,u_i\rangle \geq -1} + \ldots + \lambda_r \underbrace{\left\langle \begin{pmatrix} m \\ 1 \end{pmatrix}, \begin{pmatrix} u_r \\ 1 \end{pmatrix} \right\rangle}_{=\langle m,u_r\rangle\geq 0} \\
&\Rightarrow \begin{pmatrix} m \\ 1 \end{pmatrix} \in \sigma_P^\vee \cap (\mathbb{R}^n)^\vee \times \{1\}
\end{aligned}$$

□

4.27 Example. .

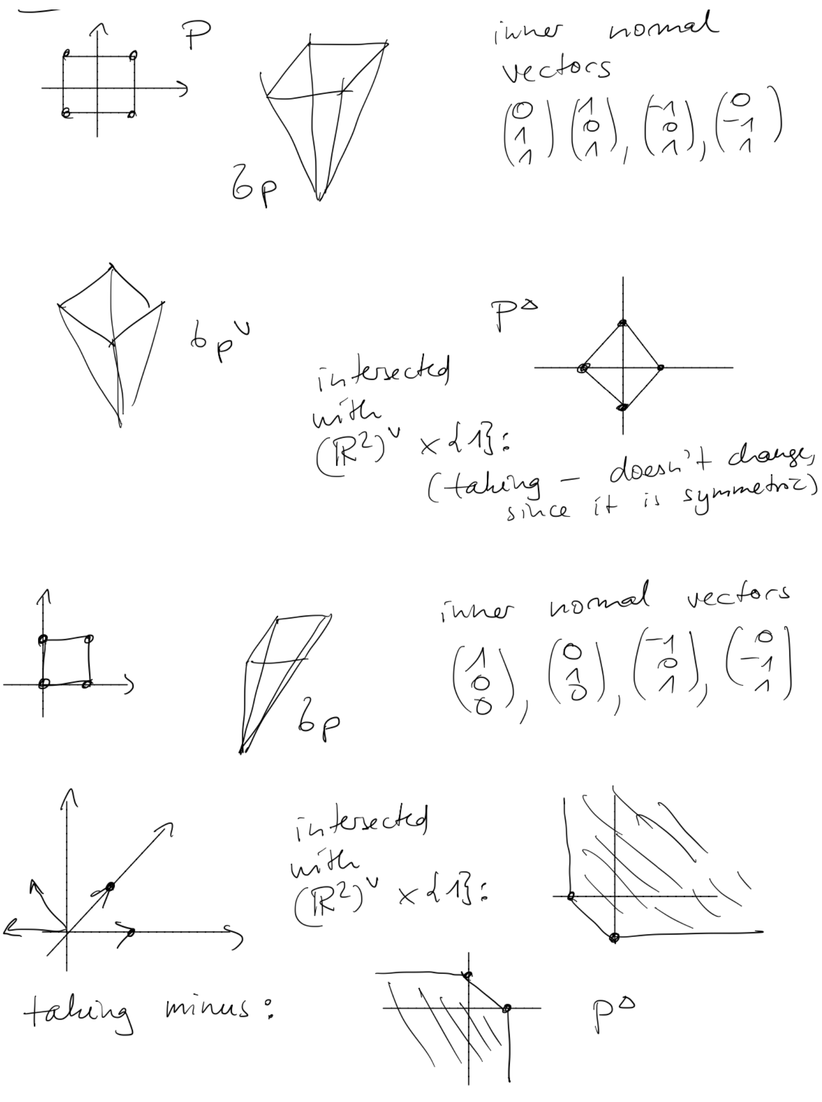


Abbildung 38: Intersections with $(\mathbb{R}^2)^\vee \times \{1\}$ [37]

Indeed, since all points in P have nonnegative coordinates, the whole negative orthant satisfies ≤ 1.

4.28 Theorem. *Let 0 be a point in the interior of P, $0 \in P^\circ$. Then $P^\triangle$ is a polytope, the polar polytope (or dual polytope), and vice versa.*

Beweis. $0 \in P$ (interior) $\Rightarrow$ all inner normal vectors of σ_P have a positive last coordinate $\Leftrightarrow \sigma_P^\vee \cap (\mathbb{R}^n)^\vee \times \{1\}$ is a polytope $\Leftrightarrow P^\triangle$ is a polytope. □

4.29 Lemma. *If $0 \in P^\circ$ then also $0 \in (P^\triangle)^\circ$.*

Beweis. $0 \in (P^\triangle)^\circ \;\Leftrightarrow \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \in (\sigma_P^\vee)^\circ.$

We can put P into a box B:

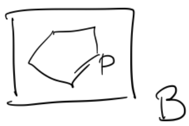


Abbildung 39: P in the box B. [38]

Then $P \subset B \Rightarrow \sigma_P \subset \sigma_B \Rightarrow \sigma_P^\vee \supset \sigma_B^\vee$.

Using lemma 4.26 and example 4.27, we see $\begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \in (\sigma_B^\vee)^\circ$. □

4.30 Theorem. *Let $0 \in P^\circ$. Then $(P^\triangle)^\triangle = P$.*

Beweis.

$$\begin{aligned} -P^\triangle &= \sigma_P^\vee \cap ((\mathbb{R}^n)^\vee \times \{1\}) \\ \Rightarrow \sigma_{-P^\triangle} &= \sigma_P^\vee \\ \Rightarrow (P^\triangle)^\triangle = -(-P^\triangle)^\triangle &= \sigma_{-P^\triangle}^\vee \cap \mathbb{R}^n \times \{1\} \\ &= (\sigma_P^\vee)^\vee \cap \mathbb{R}^n \times \{1\} \\ &= \sigma_P \cap \mathbb{R}^n \times \{1\} \\ &= P. \end{aligned}$$

□

4.31 Definition. A lattice polytope is called **reflexive** $:\Leftrightarrow$ $P^\triangle$ is a lattice polytope.

[37] Image from Hannah Markwig.
[38] Image from Hannah Markwig.

4.32 Example. $P = \operatorname{Conv}\left(\begin{pmatrix} -1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right)$
$= \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \middle| \begin{pmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \leq \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right\}$ is reflexive:

Abbildung 40: P and $P^{\triangle}$, P is reflexive. [39]

$P^{\triangle} = \operatorname{Conv}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \end{pmatrix}\right)$

4.33 Proposition. *A lattice polytope is reflexive* $\Leftrightarrow$ *it can be given in the form* $\{u|Au \leq 1\}$ *with an integer matrix* A.

Beweis. "$\Rightarrow$": From $(P^{\triangle})^{\triangle} = P$ it follows that P is reflexive. By definition,

$$P^{\triangle} = \{m \in (R^n)^{\vee} | \langle m, u \rangle \leq 1 \, \forall \, u \in P\}$$

from which we can deduce the form above, using the integer vertices u of P.
"$\Leftarrow$": Let $P = \{u|Au \leq 1\}$. Let Q be a facet of P given by the equation $\langle m, u \rangle = 1$. As A is an integer matrix, we can assume $m \in \mathbb{Z}^n$. The corresponding facet of σ_P has the equation $\left\langle \begin{pmatrix} m \\ -1 \end{pmatrix}, \begin{pmatrix} u \\ 1 \end{pmatrix} \right\rangle = 0. \Rightarrow \begin{pmatrix} m \\ -1 \end{pmatrix}$ is a ray generator of $\sigma_P^{\vee}$

$$\Rightarrow P^{\triangle} = -\sigma_P^{\vee} \cap (\mathbb{R}^n)^{\vee} \times \{1\}$$

has the vertex $m \in \mathbb{Z}^n$ and all vertices arise like this. Hence $P^{\triangle}$ is a lattice polytope. $\square$

[39] Image from Hannah Markwig.

4.34 Example. Figure 41

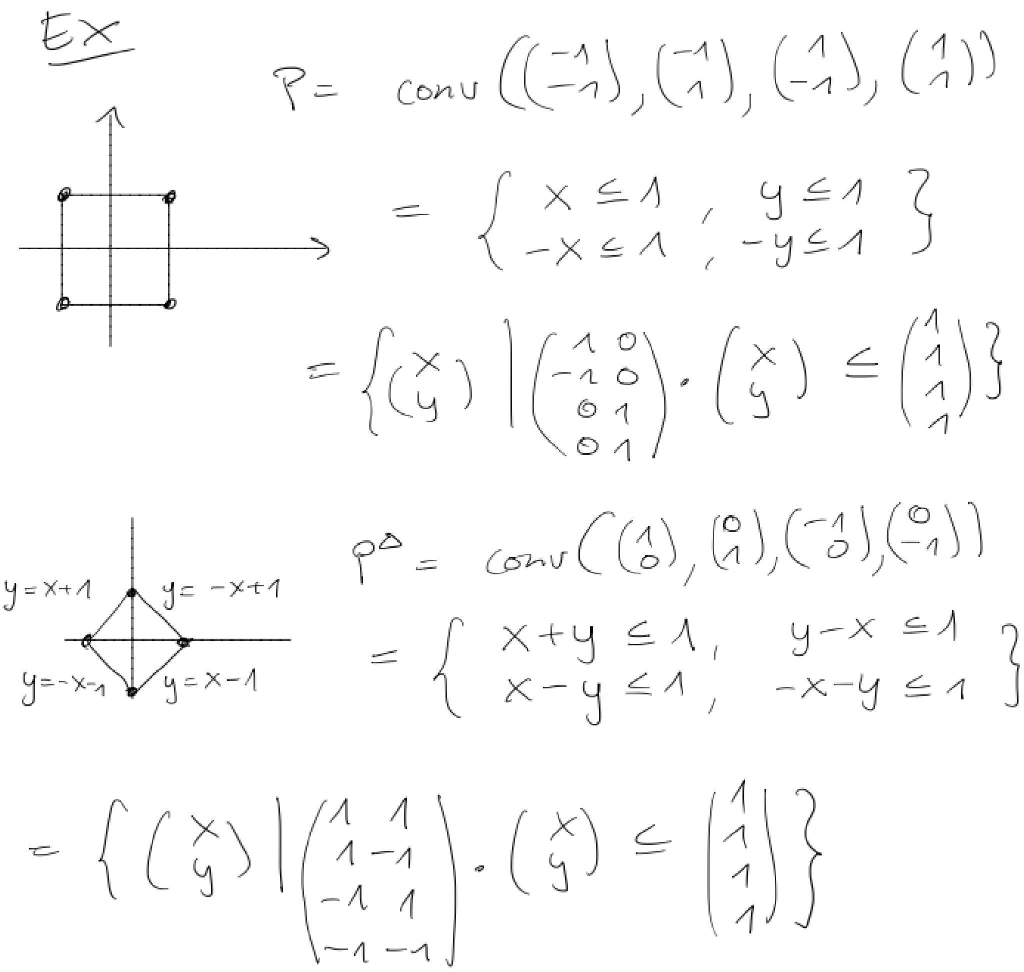


Abbildung 41: P and $P^{\triangle}$ [40]

[40]Image from Hannah Markwig.

5 Fans

5.1 Definition. Let Σ be a finite set of cones in $\mathbb{R}^n$ s.th.

(1) Every face of $\sigma \in \Sigma$ is also in Σ.

(2) The intersection $\sigma_1 \cap \sigma_2$ of $\sigma_1, \sigma_2 \in \Sigma$ is a face of σ_1 and σ_2.

Then Σ is called **fan**.
The **support** of a fan is

$$\operatorname{supp}(\Sigma) = \bigcup_{\sigma \in \Sigma} \sigma.$$

A fan is **pointed** if $0 \in \Sigma$.
A fan is **equidimensional**, if all inclusion maximal cones have the same dimension.
A fan is **complete**, if $\operatorname{supp}(\Sigma) = \mathbb{R}^n$.

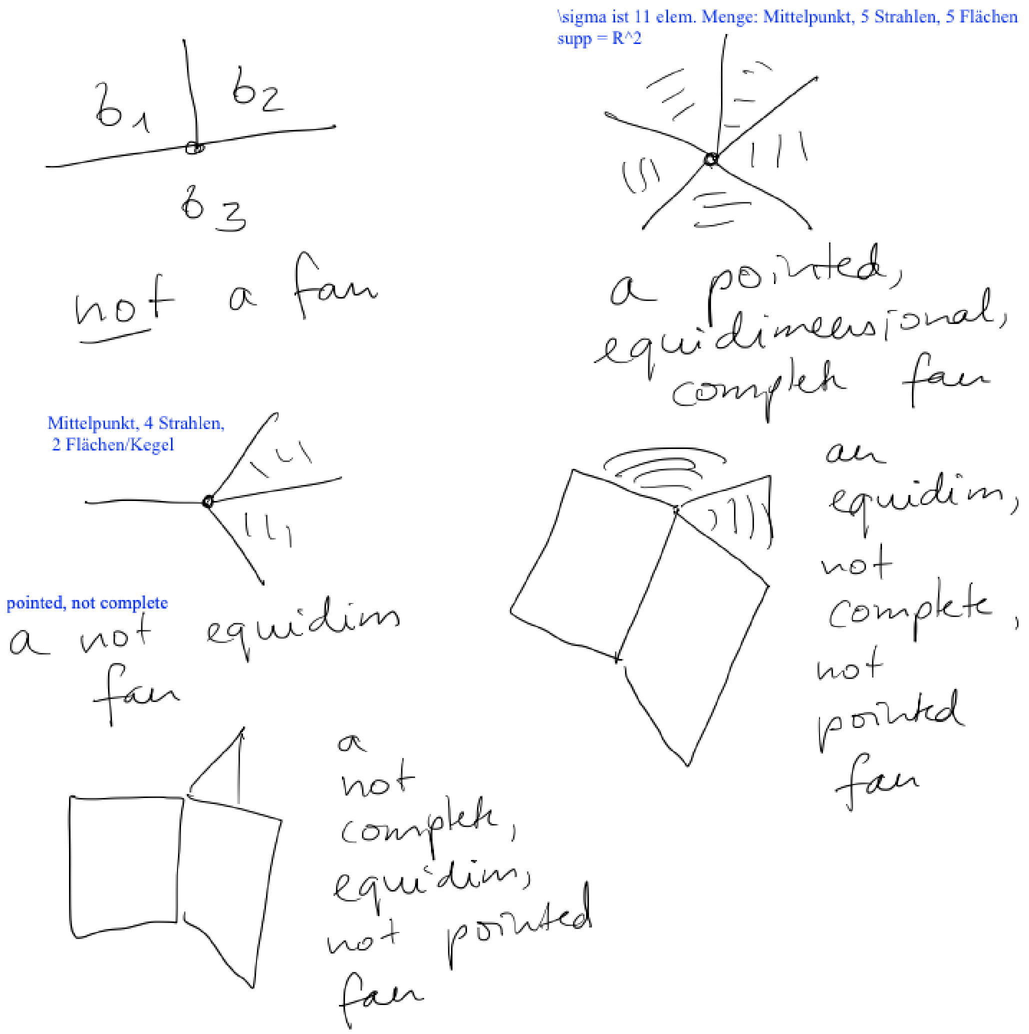


Abbildung 42: Fans [41]

[41] Image from Hannah Markwig.

We will study the (outer) normal fan of a polytope.

5.2 Definition. $P \subset \mathbb{R}^n$ a polytope, Q a face.

$$\sigma_Q := \{m \in (\mathbb{R}^n)^\vee, Q \subset \{u \in P | \langle m, u\rangle = \max_{u' \in P} \langle m, u'\rangle\}\}$$
$$\mathcal{N}(P) := \{\sigma_Q | Q \text{ face of } P\}$$

the (outer) **normal fan** of P. (Gets minimized on the face Q, minimize for the inner normal fan.)

5.3 Example. We identify $(\mathbb{R}^2)^\vee$ and $\mathbb{R}^2$.

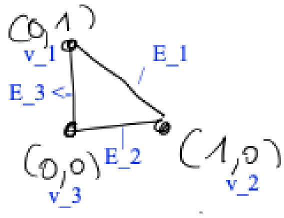


Abbildung 43: $P = \text{Conv}(v_1, v_2, v_3)$ [42]

A functional m defines an orthogonal hyperplane, it takes the same values on a shift of this hyperplane. Maximal values in P are thus taken on a face. E.g. $m = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

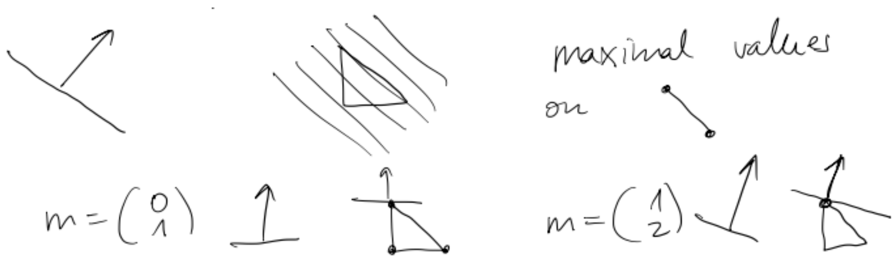


Abbildung 44: m [43]

$\sigma_{v_1} = \text{Cone}\left(\begin{pmatrix} -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right)$. $\mathcal{N}(P)$ is pictured in figure 46

5.4 Lemma. *σ_Q is a convex cone (polyhedral: later).*

Beweis. If $m \in \sigma_Q \Rightarrow$

$$\begin{aligned} Q &\subset \{u \in P | \langle m, u\rangle = \max_{u' \in P}\langle m, u'\rangle\} \\ &= \{u \in P | \lambda\langle m, u\rangle = \lambda \max_{u' \in P}\langle m, u'\rangle\} \\ &\overset{\lambda \geq 0}{=} \{u \in P | \langle \lambda m, u\rangle = \max_{u' \in P}\langle \lambda m, u'\rangle\} \end{aligned}$$

[42] Image from Hannah Markwig.
[43] Image from Hannah Markwig.
[44] Image from Hannah Markwig.

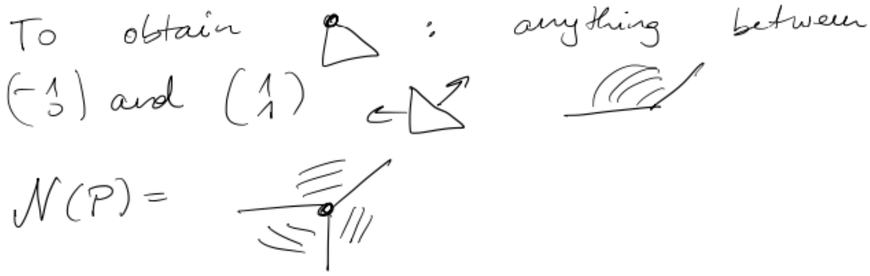


Abbildung 45: $\mathcal{N}(P)$ [44]

$\Rightarrow$ $\lambda m \in \sigma_Q$ for $\lambda > 0$. $0 \cdot m = 0 \in \sigma_Q$ since

$$Q \subset P = \{u \in P | 0 = \langle 0, u\rangle = \max_{u' \in P} \langle 0, u'\rangle\}.$$

For convexity, by the above it is sufficient to show $m_1 + m_2 \in \sigma_Q$ for $m_1, m_2 \in \sigma_Q$. Let $u \in Q$, then

$$\begin{aligned}
&\langle m_1, u\rangle \geq \langle m_1, u'\rangle \,\forall\, u' \in P,\\
&\langle m_2, u\rangle \geq \langle m_2, u''\rangle \,\forall\, u'' \in P\\
\Rightarrow\, &\langle m_1 + m_2, u\rangle = \langle m_1, u\rangle + \langle m_2, u\rangle \geq \langle m_1 + m_2, u'''\rangle \,\forall\, u''' \in P\\
\Rightarrow\, &Q \subset \{u \in P | \langle m_1 + m_2, u\rangle = \max_{u' \in P} \langle m_1 + m_2, u'\rangle\}\\
\Rightarrow\, &m_1 + m_2 \in \sigma_Q.
\end{aligned}$$

□

5.5 Lemma. *Assume* $\dim(P) = n$. *Then* $\sigma_Q = \mathrm{Cone}(m_F | Q \subset F)$ *where* F *denotes the facets of* P *and* m_F *the (up to positive multiple) unique outer normal vector on* F.

Beweis. "$\supset$": Let $m \in \mathrm{Cone}(m_F | Q \subset F), m = \sum_{Q \subset F} \lambda_F m_F, \lambda_F \geq 0$. Assume the affine hyperplane H_{m_F, a_F} cuts out F. Here, we take outer normal vectors, i.e. $P = \cap_{F \text{ facet}} H_{m_F, a_F}$. Let $b = \sum_{Q \subset F} \lambda_F a_F$.

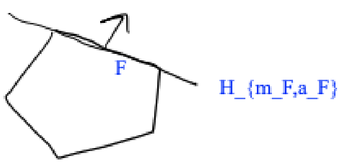


Abbildung 46: F [45]

Claim: $P \subset H^-_{m,b}$. Let $u \in P$

$$\Rightarrow \langle m, u\rangle = \langle \sum_{Q \subset F} \lambda_F m_F, u\rangle = \sum_{Q \subset F} \lambda_F \underbrace{\langle m_F, u\rangle}_{\leq a_F, \text{ since } u \in P} \leq \sum_{Q \subset F} \lambda_F a_F = b$$

[45] Image from Hannah Markwig.

Claim: $Q \subset H_{m,b} \cap P$.
$Q \subset P\checkmark$. Q is the intersection of all facets F containing Q. For any such facet, we have $u \in F \Leftrightarrow \langle m_F, u\rangle = a_F$. Hence

$$u \in Q \Leftrightarrow u \in \bigcap_{Q\subset F} F \Leftrightarrow \langle m_F, u\rangle = a_F \,\forall\, F \supset Q.$$

For $u \in Q$ we thus have

$$\langle m, u\rangle = \langle \sum_{Q\subset F} \lambda_F m_F, u\rangle = \sum_{Q\subset F} \lambda_F \underbrace{\langle m_F, u\rangle}_{=a_F} = \sum_{Q\subset F} \lambda_F a_F = b$$

$\Rightarrow u \in H_{m,b}$. Thus the functional m takes its maximum for points in P on $Q \Rightarrow m \in \sigma_Q$.
" $\subset$ ": Let $m \in \sigma_Q \Rightarrow m$ takes its maximum on Q, i.e. m defines an affine hyperplane $H_{m,b}$ with $P \subset H^-_{m,b}$ and $Q \subset H_{m,b} \cap P$. Let $v \in Q$ be a vertex of P. The facets F are given by the affine hyperplanes H_{m_F,a_F}. Consider

$$\sigma = \bigcap_{F, v\in F} H^-_{m_F,0}$$

As a finite intersection of halfspaces it is a cone, satisfying $(-\sigma)^\vee = \mathrm{Cone}(m_F | v \in F)$ by chapter 3. Since $P \subset H^-_{m,b}$ and $v \in Q \subset H_{m,b} \cap P$ we have

$$\sigma \subset H^-_{m,0} \Rightarrow -\sigma \subset H^+_{m,0} \Rightarrow m \in (-\sigma)^\vee \Rightarrow m = \sum_{F, v\in F} \lambda_F m_F, \quad \lambda_F \geq 0$$

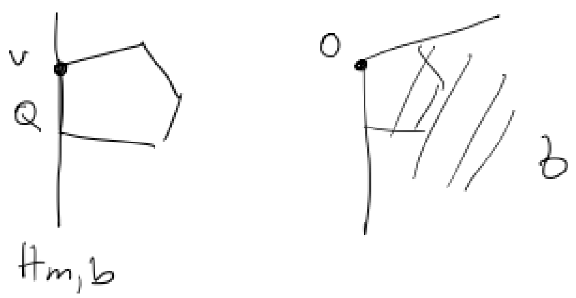


Abbildung 47: σ [46]

(Pretend that v is zero, look at the cone from the sides.) It remains to show that $\lambda_F = 0 \,\forall\, F$ s.th. $Q \not\subset F$.
Choose $p \in Q$, $p \notin F_j$ for some $Q \not\subset F_j$. Since p and $v \in Q \subset H_{m,b}$ we have

$$\begin{aligned} b = \langle m, p\rangle &= \sum_{v\in F} \lambda_F \underbrace{\langle m_F, p\rangle}_{\leq a_F} \\ b = \langle m, v\rangle &= \sum_{v\in F} \lambda_F \langle m_F, v\rangle \\ &= \sum_{v\in F} \lambda_F a_F \end{aligned}$$

[46]Image from Hannah Markwig.

where the last equality holds since $v \in F$.
Since $p \notin F_j$ we have $\langle m_{F_j}, p\rangle < a_{F_j}$. Since

$$\langle m_F, p\rangle \leq a_F \,\forall\, F\,(p \in P) \quad \text{and since} \quad \sum_{v\in F} \lambda_F \langle m_F, p\rangle = \sum_{v\in F} \lambda_F a_F$$

we conclude $\lambda_{F_j} = 0$. Hence $m = \sum_{Q\subset F} \lambda_F m_F$. □

5.6 Definition. Let τ be a face of a cone σ.

$$\tau^\perp := \{m \in (\mathbb{R}^n)^\vee | \langle m, u\rangle = 0 \,\forall\, u \in \tau\}$$
$$\tau^* := \{m \in \sigma^\vee | \langle m, u\rangle = 0 \,\forall\, u \in \tau\} = \sigma^\vee \cap \tau^\perp$$

We call τ^* the **dual face** of τ.

5.7 Example. $\sigma^* = \sigma^\vee \cap \sigma^\perp = \sigma^\vee \cap \{0\} =$ vertex
$(\text{vertex})^* = \sigma^\vee \cap \text{ vertex}^\perp = \sigma^\vee \cap (\mathbb{R}^n)^\vee = \sigma^\vee$

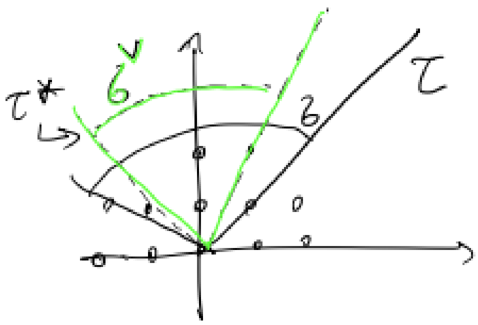


Abbildung 48: τ^* [47]

5.8 Proposition. *(1) τ^* is a face of $\sigma^\vee$.*

(2) $\dim \tau^* + \dim \tau = n$.

(3) The map $\tau \mapsto \tau^*$ is an inclusion reversing bijection between faces of σ and faces of $\sigma^\vee$.

Beweis. (1) Faces of $\sigma^\vee$ are, by definition, of the form $\sigma^\vee \cap H_v$ for some $v \in \sigma = (\sigma^\vee)^\vee$.
Let τ be the face of σ s.th. $v \in \text{Relint}(\tau)$, then

$$\sigma^\vee \cap H_v \overset{\text{as } \sigma^\vee \subset \tau^\vee}{=} (\sigma^\vee \cap \tau^\vee) \cap H_v = \sigma^\vee \cap (\tau^\vee \cap H_v) \tag{1}$$

Claim: $\tau^\vee \cap H_v = \tau^\perp$:
" $\subset$ ": for $m \in \tau^\vee$ we have $\langle m, u\rangle \geq 0 \,\forall\, u \in \tau$. Assume $\tau = \text{Cone}(v_1, \ldots, v_r)$,
$\Rightarrow \exists\, \lambda_i > 0$ s.th. $v = \sum \lambda_i v_i$, as $v \in \text{Relint}(\tau)$ $\Rightarrow$ for $m \in \tau^\vee \cap H_v$ we have

$$0 = \langle m, v\rangle = \langle m, \sum \lambda_i v_i\rangle = \sum \underbrace{\lambda_i}_{>0} \underbrace{\langle m, v_i\rangle}_{\geq 0 \text{ as } m\in\tau^\vee \text{ and } v_i\in\tau}$$
$$\Rightarrow \langle m, v_i\rangle = 0 \,\forall\, i = 1, \ldots, r \;\Rightarrow\; m \in \tau^\perp$$

" $\supset$ ": $\tau^\perp \subset \tau^\vee$. As $v \in \tau$ we have $\langle m, v\rangle = 0 \,\forall\, m \in \tau^\perp \Rightarrow \tau^\perp \subset H_v$.
Thus 1 $\sigma^\vee \cap H_v = \sigma^\vee \cap (\tau^\vee \cap H_v) = \sigma^\vee \cap \tau^\perp = \tau^*$.
Hence τ^* is a face of $\sigma^\vee$ and every face of $\sigma^\vee$ is of the form τ^*.

[47] Image from Hannah Markwig.

(3) If

$$\tau_1 \subset \tau_2 \Rightarrow \tau_1^\perp \subset \tau_2^\perp \Rightarrow \tau_1^\perp \cap \sigma^\vee \subset \tau_2^\perp \cap \sigma^\vee \Rightarrow \tau_1^* \supset \tau_2^*.$$

We have $\tau \subset (\tau^*)^*$, since

$$(\tau^*)^* = \{u \in \sigma | \langle m, u\rangle = 0 \,\forall\, m \in \tau^*\}.$$

Since inclusion is reversed, we obtain $\tau^* \supset ((\tau^*)^*)^*$.
But if we insert the face τ^* of $\sigma^\vee$ in the relation $\tau \subset (\tau^*)^*$ above, we obtain $\tau^* \subset ((\tau^*)^*)^* \Rightarrow \tau^* = ((\tau^*)^*)^*$. Thus the map $\tau \mapsto \tau^*$ is a bijection.

(2) The smallest face of σ is

$$(\sigma^\vee)^* = (\sigma^\vee)^\perp \cap (\sigma^\vee)^\vee = (\sigma^\vee)^\perp \cap \sigma$$

Notice $(\sigma^\vee)^\perp \subset \sigma$ since

$$\begin{aligned}(\sigma^\vee)^\perp &= \{u \in \mathbb{R}^n | \langle m, u\rangle = 0 \,\forall\, m \in \sigma^\vee\} \\ &\subset \{u \in \mathbb{R}^n | \langle m, u\rangle \geq 0 \,\forall\, m \in \sigma^\vee\} \\ &= (\sigma^\vee)^\vee = \sigma\end{aligned}$$

$$\Rightarrow (\sigma^\vee)^* = (\sigma^\vee)^\perp \cap \sigma = (\sigma^\vee)^\perp = \sigma \cap (-\sigma),$$

since

$$\left.\begin{aligned} u \in \sigma &\Leftrightarrow \langle m, u\rangle \geq 0 \,\forall\, m \in \sigma^\vee \\ u \in -\sigma &\Leftrightarrow \langle m, u\rangle \leq 0 \,\forall\, m \in \sigma^\vee \end{aligned}\right\} \Rightarrow$$
$$u \in \sigma \cap (-\sigma) \Leftrightarrow \langle m, u\rangle = 0 \,\forall\, m \in \sigma^\vee \Leftrightarrow u \in (\sigma^\vee)^\perp$$
$$\Rightarrow \dim \sigma^\vee + \dim(\sigma^\vee)^* = \dim \sigma^\vee + \dim(\sigma^\vee)^\perp = n$$

Analogously, the smallest face of $\sigma^\vee$ is σ^* and $\dim \sigma + \dim \sigma^* = n$. Given a face τ of σ, $\tau \subset \sigma$, consider a maximal chain of faces of σ containing τ:

$$\underset{=(\sigma^\vee)^*}{\tau_0} \subset \tau_1 \subset \ldots \subset \underset{=\tau}{\tau_i} \subset \ldots \subset \tau_r = \sigma$$

We have $\dim \tau_j = \dim(\sigma^\vee)^* + j$ and $\dim(\sigma^\vee)^* + r = \dim \sigma$. Consider the dual chain:

$$\begin{array}{cccccc} \sigma^\vee & \supset \tau_1^* & \supset \ldots & \supset \underset{=\tau^*}{\tau_i^*} & \supset \ldots & \supset \sigma^* \\ n - \dim \sigma + r & n - \dim \sigma + r - 1 & \ldots & n - \dim \sigma + r - i & \ldots & n - \dim \sigma \\ \underset{=(\sigma^\vee)^*}{\tau_0} & \subset \tau_1 & \subset \ldots & \subset \underset{=\tau}{\tau_i} & \subset \ldots \subset \tau_r & = \sigma \\ \dim \sigma - r & \dim \sigma - r + 1 & \ldots & \dim \sigma - r + i & \ldots & \dim \sigma \end{array}$$

$$\Rightarrow \dim \tau^* + \dim \tau = n - \dim \sigma + r - i + \dim \sigma - r + i = n$$

$\square$

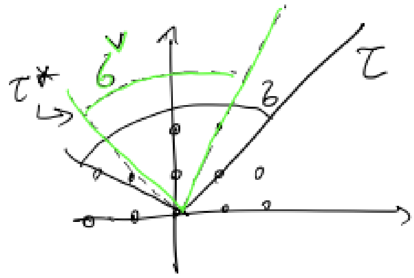


Abbildung 49: τ^* [48]

5.9 Example. $v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \in \mathrm{Relint}(\tau)$, $\sigma^\vee \cap H_v = \tau^*$, $v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \in \mathrm{Relint}(\sigma)$,
$\sigma^* = \sigma^\vee \cap \sigma^\perp = \sigma^\vee \cap \{0\} = \text{vertex} = \sigma^\vee \cap H_{v_2}$, $v_3 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \in \mathrm{Relint(vertex)}$,
$(\text{vertex})^* = \sigma^\vee \cap \text{vertex}^\perp = \sigma^\vee \cap (\mathbb{R}^n)^\vee = \sigma^\vee = \sigma^\vee \cap H_{v_3}$

5.10 Theorem (Normal fan)**.** *Let P be a fulldimensional polytope.*
Let Q, Q' be faces of P.

(1) $Q \subset Q' \Leftrightarrow \sigma_Q \supset \sigma_{Q'}$

(2) If $Q \subset Q'$, $\sigma_{Q'}$ is a face of σ_Q and all faces of σ_Q are of this form.

(3) $\sigma_Q \cap \sigma_{Q'} = \sigma_{Q''}$, where Q'' is the smallest face of P containing Q and Q'.

Beweis. (1) " $\Rightarrow$ ":

$$\sigma_{Q'} = \mathrm{Cone}(m_F | Q' \subset F) \underset{\uparrow}{\subset} \mathrm{Cone}(m_F | Q \subset F) = \sigma_Q$$

since every facet containing Q' also contains Q.
" $\Leftarrow$ ": If $\sigma_{Q'} \subset \sigma_Q \Rightarrow \forall$ generators m_F of $\sigma_{Q'}$ (i.e. $\forall\, m_F$ with $Q' \subset F$) we also have $m_F \in \sigma_Q \Rightarrow \forall$ facets F with $Q' \subset F\, \exists$ supporting hyperplane H_{m_F, a_F} of P s.th.

$$Q \subset F = H_{m_F, a_F} \cap P \Rightarrow Q \subset \bigcap_{F | Q' \subset F} F = Q'$$

(2) Let $v \in Q$ be a vertex. As before we consider $\sigma = \cap_{F|v \in F} H^-_{m_F, 0}$

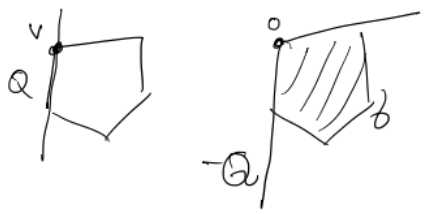


Abbildung 50: Q and $\mathcal{Q}$ [49]

[48]Image from Hannah Markwig.

Q corresponds to a face $\mathcal{Q}$ of $(-\sigma)$ which, by proposition 5.10 yields the dual face $\mathcal{Q}^* = (-\sigma)^\vee \cap \mathcal{Q}^+$ of $(-\sigma)^\vee$.
We have $(-\sigma)^\vee = \text{Cone}(m_F | v \in F) = \sigma_v$ and $\mathcal{Q}^* = \text{Cone}(m_F | v \in F$ and $\mathcal{Q} \subset H_{m_F,0})$.
As $v \in Q$, we can conclude from $\mathcal{Q} \subset H_{m_F,0}, Q \subset H_{m_F,a_F} \Rightarrow Q \subset F$
$\Rightarrow \mathcal{Q}^* = \text{Cone}(m_F | Q \subset F) \Rightarrow \sigma_Q$ is the dual face to $\mathcal{Q}$ of $(-\sigma)^\vee = \sigma_v$. Since all faces of $(-\sigma)^\vee$ are of the form $\mathcal{Q}^*$ for some $\mathcal{Q}$, also all faces of σ_v are of the form σ_Q. With that, also all faces of σ_Q are of the form σ'_Q.
If $Q \subset Q'$, $\sigma_{Q'}$ is also a face of $(-\sigma)^\vee = \sigma_v$, and hence because of 1) $\sigma_{Q'}$ is a face of σ_Q.

(3) Let Q'' be the smallest face of P containing Q and Q'. Then

$$Q'' = \bigcap_{F|Q'' \subset F} F = \bigcap_{\substack{F|Q \subset F \\ \text{and } Q' \subset F}} F$$

(if the intersection is empty, $Q'' = P$)
By 2), $\sigma_{Q''}$ is a face of σ_Q and of

$$\sigma_{Q'} \Rightarrow \sigma_{Q''} \subset \sigma_Q \cap \sigma_{Q'} \tag{2}$$

Assume $\sigma_Q \cap \sigma_{Q'} = \{0\} = \sigma_P$, then

$$\sigma_Q \cap \sigma_{Q'} \subset \sigma_{Q''}, \sigma_{Q''} = \{0\} = \text{Cone}(m_F | Q'' \subset F) = \text{Cone}(\emptyset) \Rightarrow Q'' = P.$$

Assume now $\sigma_Q \cap \sigma_{Q'} \neq \{0\}$ and let $0 \neq m \in \sigma_Q \cap \sigma_{Q'}$.
Let $b = \max_{v \text{ vertex of } P} \langle m, v \rangle$. Claim: $P \subset H^-_{m,b}$.
If

$$u \in P \Rightarrow u = \sum \lambda_v v, \lambda_v \geq 0, \sum \lambda_v = 1. \langle m, u \rangle = \sum \lambda_v \langle m, v \rangle \leq \sum \lambda_v b = (\sum \lambda_v) b = b$$

Since $m \in \sigma_Q, \sigma_{Q'}$ and $H_{m,b}$ is a supporting hyperplane of $P \Rightarrow$ the form m takes its maximum on Q and on $Q' \Rightarrow Q, Q' \subset H_{m,b} \cap P$. Since $H_{m,b} \cap P$ is a face that contains Q and Q', and Q'' is the smallest face that contains Q and Q'

$$\begin{aligned}
&\Rightarrow Q'' \subset H_{m,b} \cap P \\
&\Rightarrow m \text{ takes its maximum on } Q'' \\
&\Rightarrow m \in \sigma_{Q''} \\
&\Rightarrow \sigma_Q \cap \sigma_{Q'} \subset \sigma_{Q''} \\
&\overset{(2)}{\Rightarrow} \sigma_Q \cap \sigma_{Q'} = \sigma_{Q''}
\end{aligned}$$

□

5.11 Definition. The **lineality space** of a fan Σ is the biggest subspace of $\mathbb{R}^n$ which is contained in every cone of Σ.

5.12 Example.
- Σ pointed $\Leftrightarrow$ lineality space $= \{0\}$

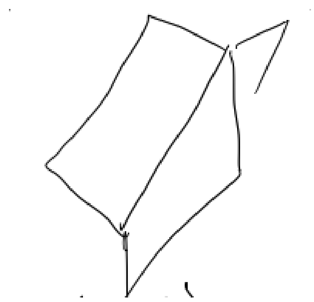


Abbildung 51: Σ [50]

- $\Sigma = \left\{ \text{Cone}\left(\begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}-1\\-1\\-1\end{pmatrix}, \begin{pmatrix}1\\0\\0\end{pmatrix}\right), \text{Cone}\left(\begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}-1\\-1\\1\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}\right), \right.$
  $\left. \text{Cone}\left(\begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}-1\\-1\\-1\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\right), \text{Cone}\left(\begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}-1\\-1\\-1\end{pmatrix}\right) \right\}$

  Lineality space $= \left\langle \begin{pmatrix}1\\1\\1\end{pmatrix} \right\rangle$

5.13 Definition. Let Σ be a fan and $\sigma \in \Sigma$. The **star of** $\sigma \in \Sigma$ is a fan in $\mathbb{R}^n$, $\text{star}_\Sigma(\sigma)$, whose cones are indexed by $\tau \in \Sigma$ with σ being a face of τ. Let $w \in \sigma$. The cone in $\text{star}_\Sigma(\sigma)$ with index τ is the Minkowski sum

$$\overline{\tau} = \{v \in \mathbb{R}^n | \exists \epsilon > 0 : w + \epsilon v \in \tau\} + \text{span}\,\sigma - w$$

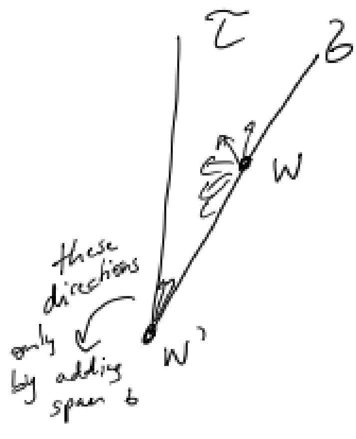


Abbildung 52: w and w' [51]

[49] Image from Hannah Markwig.
[50] Image from Hannah Markwig.
[51] Image from Hannah Markwig.

We show that $\overline{\tau}$ is independent of the choice of w:

$$\{v \in \mathbb{R}^n | \exists\, \epsilon > 0 : w + \epsilon v \in \tau\} + \operatorname{span} \sigma - w$$
$$\subset \{v' \in \mathbb{R}^n | \exists\, \epsilon' > 0 : w' + \epsilon' v' \in \tau\} + \operatorname{span} \sigma - w' :$$

Let $v + r - w$ be in the left hand side, i.e. $w + \epsilon v \in \tau$ for some $\epsilon > 0$, $r \in \operatorname{span}(\sigma)$. Let $v' = \frac{1}{\epsilon}(\epsilon v + w - w')$. Then

$$w' + \epsilon v' = w' + \epsilon v + w - w' = \epsilon v + w \in \tau.$$
$$v + r - w = \underbrace{\frac{1}{\epsilon}(\epsilon v + w - w')}_{v'} + \underbrace{(-\frac{1}{\epsilon})(w - w') + r + w' - w - w'}_{\in \operatorname{span}(\sigma)}$$

is contained in the right hand side.

5.14 Example. The lineality space of $\operatorname{star}_\Sigma(\sigma)$ is $\operatorname{span}(\sigma)$.

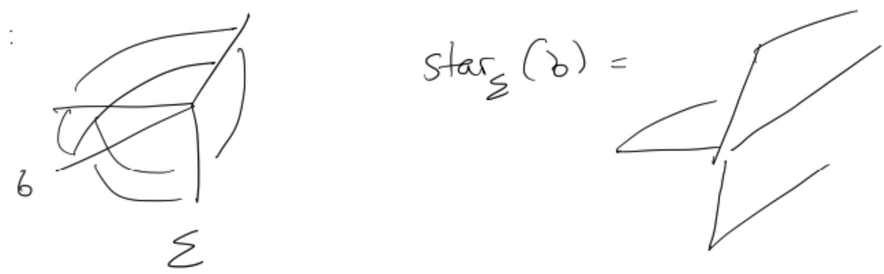


Abbildung 53: $\operatorname{star}_\Sigma(\sigma)$ [52]

<u>Exercise:</u> Let $Q \subset P$ be a face, then $\mathcal{N}(Q) = \operatorname{star}_{\mathcal{N}(P)}(\sigma_Q)$ and has lineality space $\operatorname{span}(\sigma_Q) = \operatorname{span}(Q)^\perp$. In particular, normal fans of of polytopes which are not full-dimensional can be viewed as the sum of the normal fan in its span and a lineality space.

5.15 Example. .

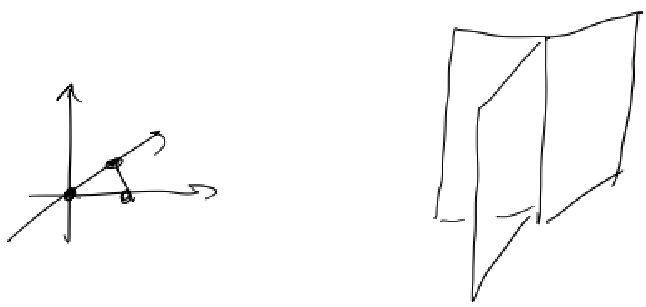


Abbildung 54: $\operatorname{star}_\Sigma(\sigma)$ [53]

[52]Image from Hannah Markwig.
[53]Image from Hannah Markwig.

5.16 Proposition. *Let $P \subset \mathbb{R}^n$ be full-dimensional, $\mathcal{N}(P)$ its normal fan. Then:*

(1) $\dim Q + \dim \sigma_Q = n$

(2)

$$(\mathbb{R})^\vee = \bigcup_{v \text{ vertex of } P} \sigma_v = \bigcup_{Q \text{ face of } P} \sigma_Q$$

In particular, $\mathcal{N}(P)$ is complete.

Beweis. (1) Let v be a vertex of Q. As before, Q corresponds to a face $\mathcal{Q}$ of $\cap_{F, v \in F} H^-_{m_F,0}$ and for the dual face $\mathcal{Q}^*$ of the dual cone $\sigma_v = \text{Cone}(m_F | v \in F)$ we have $\mathcal{Q}^* = \sigma_Q$. Then

$$\dim Q + \dim \sigma_Q = \dim \mathcal{Q} + \dim \mathcal{Q}^* = n.$$

(2) Let $m \in (\mathbb{R}^n)^\vee$. Let $b = \max\{\langle m, v\rangle | v \text{ vertex of } P\}$. Then $P \subset H^-_{m,b}$ as we saw before and $v \in H_{m,b}$ for at least one vertex v of P. Then m takes its maximum at $v \Rightarrow m \in \sigma_v$. The second equality is clear.

□

Not every complete fan is the normal fan of a polytope, see exercises. Our next goal is to study normal fans of lattice polygons. We will see that there is a natural way to define weights in their codim$-1-$skeleta s.th. they fulfill the so-called balancing conditions. To define these weights and to state the balancing condition, we need to discuss **lattice indices**.

More about this topic can be studied in an algebra class or any class that discusses the classification of finitely general abelian groups.

Remember: for a (finite) group G and a subgroup U, we call $|G/U|$ the **index**. If $G = \mathbb{Z}$ and $U = m\mathbb{Z}$, $|\mathbb{Z}/m\mathbb{Z}| = m$

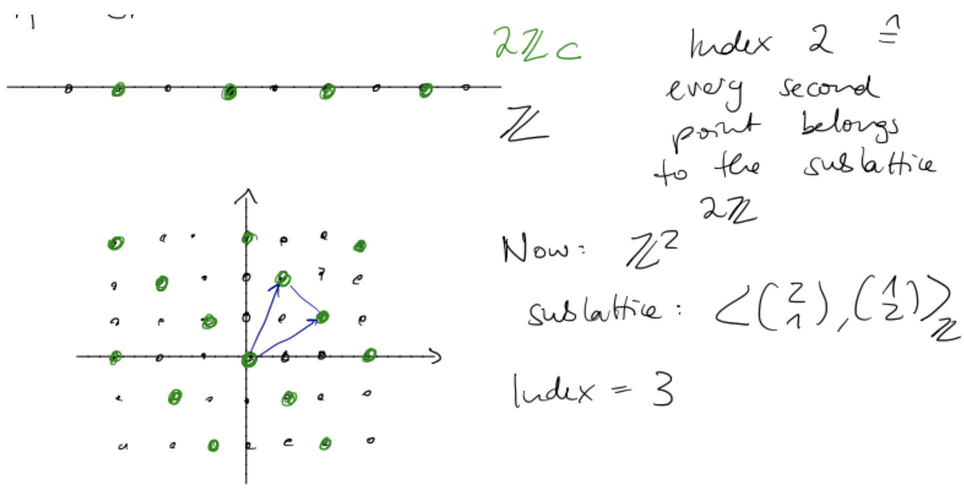


Abbildung 55: Index 2 and 3. [54]

[54]Image from Hannah Markwig.

How to understand the $\mathbb{Z}$-span of $\begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix}$?

$\mathbb{Z}^n$ is a module (like a vector space over a ring instead of a field). Modules are generally more difficult than vector spaces, since they do not need to have a basis. But $\mathbb{Z}^n$ has a basis: $e_1, \ldots, e_n$. Every point in $\mathbb{Z}^n$ can be uniquely written as a $\mathbb{Z}$-linear combination of the e_i. Therefore it is not much different to work with $\mathbb{Z}^n$ than with a vector space. $\left\langle \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\rangle_{\mathbb{Z}}$ denotes the $\mathbb{Z}$-linear hull. One can also view it as the subgroup generated by $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ in $\mathbb{Z}^2$. Every abelian group is a $\mathbb{Z}$-module, if we define

$$\begin{aligned} z \cdot g &:= \underbrace{g + \ldots + g}_{z \text{ times}} \quad \text{if} \quad z > 0, \\ z \cdot g &:= \underbrace{-g - \ldots - g}_{|z| \text{ times}} \quad \text{if} \quad z < 0, \end{aligned}$$

and $0 \cdot g := 0$. The $\mathbb{Z}$-linear structure is thus implicitly given by the structure as abelian group.

5.17 Theorem (Elementary divisors)**.** *Let $U \subset \mathbb{Z}^n$ be a submodule, then there exists a basis $v_1, \ldots, v_n$ of $\mathbb{Z}^n$ and a basis $(u_1, \ldots, u_m)$ of U s.th. $\lambda_i \cdot v_i = u_i \mod \langle v_1, \ldots, v_{i-1} \rangle$. The λ_i are (up to sign) uniquely determined by U and are called the* ***elementrary divisiors*** *of U.*

Beweis. See classes on Algebra. □

Idea: Given generators of U, write them into the rows of a matrix. To get a nice form, we would usually apply the Gauß Algorithm to get it to row echelon form - then we get a relation of our new set of generators to the vectors in the canonical basis, e.g.

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \end{pmatrix} \quad e_1 + 2e_3,\ e_2 + 3e_3$$

But when we do Gauß we have to divide! We can't do that here, over $\mathbb{Z}$.
Ex:

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \xrightarrow{\mathrm{II} \mapsto \mathrm{II} - 2\mathrm{I}} \begin{pmatrix} 1 & 2 \\ 0 & -3 \end{pmatrix}$$

now we can't continue since we cannot divide by 3. At least we got a triangular form. In general, one can obtain diagonal form: with the Euclidean algorithm, for matrix entries a, b satisfying $\gcd(a, b) = 1$, we obtain an expression of the form $ua + vb = 1$ which we can use to produce pivots. If some gcd are not 1, we obtain (non-trivial) elementary divisors λ_i. In the end, one obtains the so-called **Smith-Normal-Form** of an integer matrix, which is diagonal and has the elementary divisors λ_i on its diagonal:

$$\left(\begin{array}{ccc|c} \lambda_1 & & & \\ & \ddots & & * \\ & & \lambda_m & \end{array} \right)$$

For more details on the algorithm to produce a Smith-Normal-Form, see class on Algebra.

5.18 Corollary. *$\mathbb{Z}^n/U$ is the direct sum of a free module with summands of the form $\mathbb{Z}^n/\lambda_i\mathbb{Z}$:*

$$\mathbb{Z}^n/U = \mathbb{Z}^{n-m} \times \mathbb{Z}/\lambda_i\mathbb{Z} \times \ldots \times \mathbb{Z}/\lambda_m\mathbb{Z}$$

Idea:

$$\begin{aligned}\mathbb{Z}^n &= \langle e_1, \ldots, e_n\rangle_{\mathbb{Z}} = \langle v_1, \ldots, v_n\rangle_{\mathbb{Z}} \\ &= \langle v_1\rangle \times \ldots \times \langle v_n\rangle,\ U = \langle \lambda_1 v_1\rangle \times \ldots \times \langle \lambda_m v_m\rangle \\ &\Rightarrow \mathbb{Z}^n/U = \langle v_1\rangle/\langle \lambda_1 v_1\rangle \times \ldots \times \langle v_m\rangle/\langle \lambda_m v_m\rangle \times \langle v_{m+1}, \ldots, v_n\rangle \\ &= \mathbb{Z}/\lambda_1\mathbb{Z} \times \ldots \times \mathbb{Z}/\lambda_m\mathbb{Z} \times \mathbb{Z}^{n-m}\end{aligned}$$

If $U \subset \mathbb{Z}^n$ is a submodule of full rank ($n = m$), $\mathbb{Z}^n/U$ is a finite group. Then:

5.19 Definition. $|\mathbb{Z}^n/U|$ is the **lattice index** of U in $\mathbb{Z}^n$.
Using theorem 5.17, it equals

$$|\prod_{i=1}^{n} \lambda_i|.$$

Given a basis of U in coordinates, write it into a matrix (square, size n). Since we use only $\mathbb{Z}$-row-operations to bring it to Smith-Normal-Form

$$\begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_m \end{pmatrix},$$

the absolute value of its determinant equals the absolute value of the product of its elementary divisors. We can thus compute lattice indices as determinants:

5.20 Example.

$$U = \left\langle \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\rangle_{\mathbb{Z}}$$

The matrix $\begin{pmatrix} 1 & 2 \\ 0 & -3 \end{pmatrix}$ already satisfies the needs, we can let

$$v_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix},\ u_1 = -3 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix},\ v_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix},\ u_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix},$$

then $\langle u_1, u_2\rangle_{\mathbb{Z}} = U$ and $u_i = v_i \mod \langle v_1, \ldots, v_{i-1}\rangle$ as required by theorem 5.17. The elementary divisors of U are thus $1, -3$.
The lattice index $|\mathbb{Z}^2/U|$ equals $|1 \cdot (-3)| = 3$. It also equals $|\det \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}| = 3$.

5.21 Definition. Let G_i be groups, $f_i : G_i \to G_{i+1}$ group homomorphisms. The sequence

$$\ldots G_i \xrightarrow{f_i} G_{i+1} \xrightarrow{f_{i+1}} G_{i+2} \ldots$$

is called **exact** at G_{i+1} if $\ker(f_{i+1}) = \operatorname{Im}(f_i)$. In particular $f_{i+1} \circ f_i = 0$.

5.22 Example. 1) $\ldots A \xrightarrow{f} B \longrightarrow 0$ exact $\Leftrightarrow f$ surjective.
($B \longrightarrow 0$ is the zero map, $\ker = B = \mathrm{Im}(f)$).

2) $0 \longrightarrow A \xrightarrow{f} B \ldots$ exact $\Leftrightarrow f$ injective.
($\ker(f) = \mathrm{Im}(0 \to A) = \{0\}$)

5.23 Definition. A sequence $0 \longrightarrow A \xrightarrow{g} B \xrightarrow{f} C \longrightarrow 0$ which is exact everywhere is called a **short exact sequence**.

$$\Leftrightarrow \ \mathrm{Im}(g) = \ker(f),\ g \text{ injective},\ f \text{ surjective}$$

5.24 Lemma. *Let* $0 \longrightarrow A \xrightarrow{g} B \xrightarrow{f} C \longrightarrow 0$ *be a short exact sequence of finite groups. Then* $|A| \cdot |C| = |B|$.

Beweis. For $x, y \in B$ define

$$x \sim y \ :\Leftrightarrow\ f(x) = f(y) \in C.$$

In every equivalence class, we have $|\ker(f)|$ elements $\Rightarrow |B| = |B/\sim| \cdot |\ker(f)|$. But $|B/v| = |C|$ as f is surjective, and $|\ker(f)| = |\mathrm{Im}(g)| = |A|$, as g is injective. $\square$

5.25 Definition. Let Σ be an equidimensional, d-dim, $\mathbb{Z}$-rational fan in $\mathbb{R}^n$. Let $\sigma \in \Sigma$ be of top dimension, τ of dim $d-1$ and a face of σ. We call $u_{\sigma/\tau} \in \sigma$ a **normal vector** of σ w.r.t. τ, if $\exists$ lattice basis B of $\mathrm{span}(\tau)$ (i.e. a basis for $\mathbb{Z}^n \cap \mathrm{span}(\tau)$) s.th. $B \cup \{u_{\sigma/\tau}\}$ is a lattice basis of $\mathrm{span}(\sigma)$.

5.26 Example. $\tau = \langle (1,1) \rangle$. $(-1,1)$ is not a normal vector but $(0,1), (1,0), (1,2), \ldots$

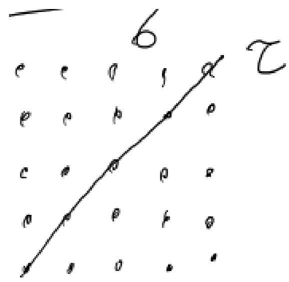


Abbildung 56: σ and τ. [55]

The normal vector of σ w.r.t. τ is not unique. If τ is the vertex and σ a ray, $U_{\sigma/\tau}$ is unique, it is the ray generator.

5.27 Definition. Let Σ be an equidimensional, d-dim, $\mathbb{Z}$-rational fan in $\mathbb{R}^n$. Let ω be a weight function on the top-dim cones, we require $\omega(\sigma) \in \mathbb{N}\, \forall \sigma$ top-dim. (Σ, ω) is called a **balanced fan** if the following **balancing condition** is satisfied for every cone τ of dimension $d-1$:

$$\sum_{\substack{\tau \subset \sigma \\ \sigma \text{ top dim}}} \omega(\sigma) \cdot u_{\sigma/\tau} = 0 \quad \text{in} \quad \mathbb{R}^n / \mathrm{span}(\tau)$$

[55] Image from Hannah Markwig.

Well-defined: If $u_{\sigma,\tau}$ and $u'_{\sigma/\tau}$ are normal vectors of σ w.r.t. τ, then $B \cup \{u_{\sigma/\tau}\}$ and $B \cup \{u'_{\sigma/\tau}\}$ are lattice basis for $\mathrm{span}(\sigma)$.
We must have $u_{\sigma/\tau} = u'_{\sigma/\tau} \mod \mathrm{span}(\tau)$ as both are lattice basis and since both are in σ.

5.28 Definition. A **refinement** of a fan Σ is a fan Σ' s.th. $\mathrm{supp}(\Sigma) = \mathrm{supp}(\Sigma')$ and $\forall\, \sigma' \in \Sigma' \,\exists\, \sigma \in \Sigma : \sigma' \subset \sigma$

5.29 Example. .

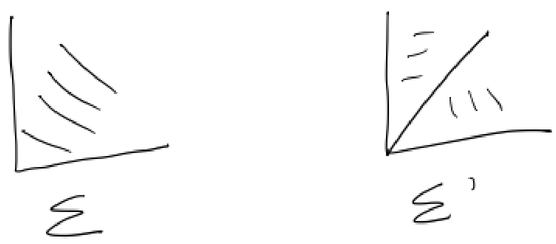


Abbildung 57: Σ and Σ'. [56]

5.30 Definition. For two fans Σ, Σ' with $\mathrm{supp}(\Sigma) = \mathrm{supp}(\Sigma')$, the **common refinement** is $\{\sigma \cap \sigma' | \sigma \in \Sigma, \sigma' \in \Sigma'\}$. It is a refinement of both.

5.31 Example. .

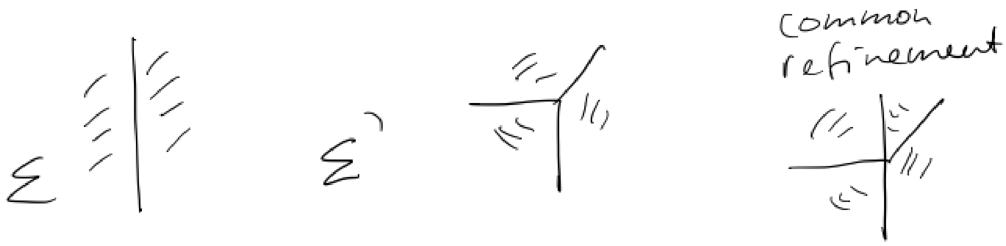


Abbildung 58: Common refinement of Σ and Σ'. [57]

5.32 Definition. Two weighted fans (Σ, ω), (Σ', ω') with $\mathrm{supp}(\Sigma) = \mathrm{supp}(\Sigma')$ are called **equivalent**, if their common refinement respects the weights.

5.33 Example. Figure 59

5.34 Remark. Being balanced depends only on the equivalence class, not on a representative. If we subdivide a top-dim cone by a codim-1-wall, the two new normal vectors can be chosen opposite to each other, and the weight is the same, so the balancing condition on the new codim-1-cone reads $\omega \cdot u - \omega \cdot u = 0$. We thus often consider balanced fans only up to equivalence, see Figure 60.

[56]Image from Hannah Markwig.
[57]Image from Hannah Markwig.
[58]Image from Hannah Markwig.
[59]Image from Hannah Markwig.

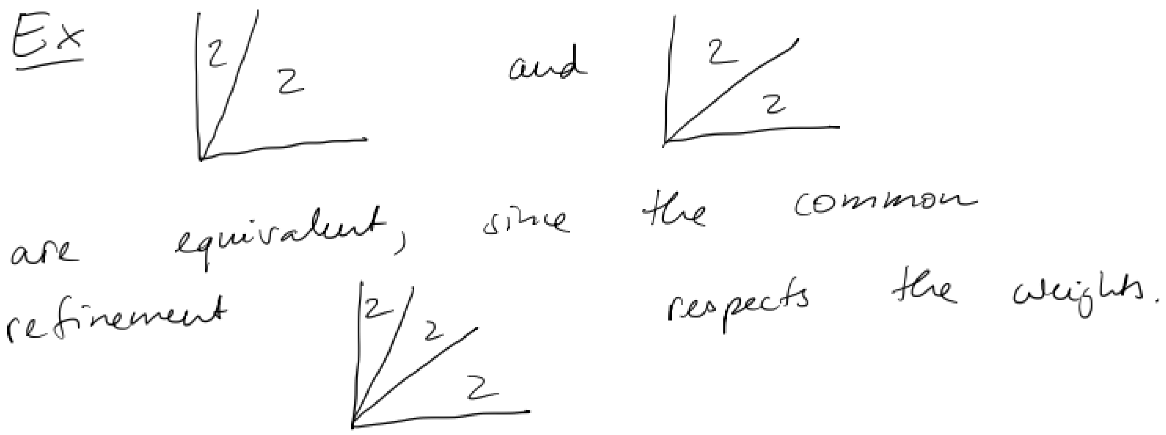


Abbildung 59: Common refinement of Σ and Σ' respecting the weights. [58]

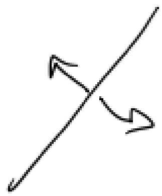


Abbildung 60: Balancing condition [59]

5.35 Definition (1)).

1. Let Σ be an equidimensional fan. We denote by $\Sigma^{(k)}$ the **codim-k-skeleton** of Σ, i.e. the set of all codim-k-cones in Σ and their faces.
2. Let $\Sigma = \mathcal{N}(P)$ be the normal fan of a top-dim lattice polygon. A top-dim cone σ_Q of $\Sigma^{(1)} \mathrel{\widehat{=}}$ a $(n-1)$-dim cone of $\Sigma \mathrel{\widehat{=}}$ a 1-dim face of Q, i.e. an edge

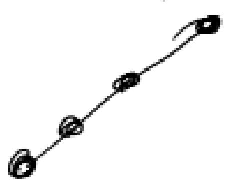


Abbildung 61: Edge [60]

We define $\omega(\sigma_Q) :=$ lattice length of the edge $Q = |Q \cap \mathbb{Z}^n| - 1$. $(\Sigma^{(1)}, \omega)$ is called the **(standard) hyperplane** dual to P.

5.36 Example. Figure 62

[60]Image from Hannah Markwig.
[61]Image from Hannah Markwig.

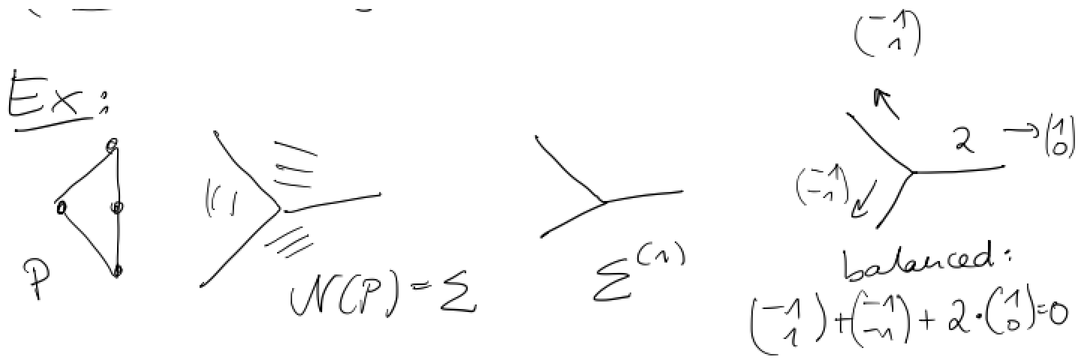


Abbildung 62: Standard hyperplane [61]

5.37 Theorem. *The hyperplane dual to P is a balanced fan.*

Beweis. Let τ be a cone if codim 1 of $\mathcal{N}(P)^{(1)}$, i.e. a cone of codim 2 in $\mathcal{N}(P)$, corresponding to a 2-dim face of Q of P. Let $K_1, \ldots, K_r$ be the edges of Q.

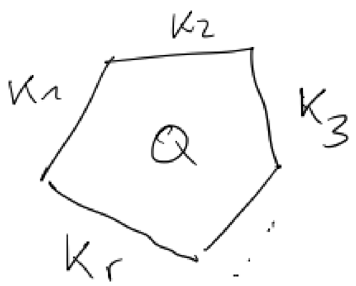


Abbildung 63: Q [62]

The top-dim cones neighbouring τ in $\mathcal{N}(P)^{(1)}$ correspond to the edges $K_1, \ldots, K_r$, call them $\sigma_1, \ldots, \sigma_r$. We can check the balancing condition around τ at the star, $\mathrm{star}_{\mathcal{N}(P)^{(1)}}(\tau)$. By the exercise, $\mathrm{star}_{\mathcal{N}(P)^{(1)}} = \mathcal{N}(Q)^{(1)} \times \mathrm{span}(Q)^{\perp}$.
It remains to show: the hyperplane of Q is balanced. The ray generators of the $\overline{\sigma_i}$ are the outer normal vector of the edges of Q in $\mathrm{span}(Q) = \mathbb{R}^2$.

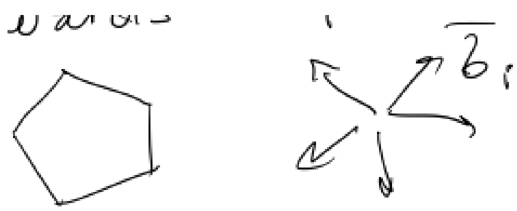


Abbildung 64: $\overline{\sigma_i}$ [63]

Weighted with the lattice length of their dual edges, they sum up to 0, because the dual edges form a closed path in $\mathbb{R}^2$ (the boundary of the polygon Q). □

[62]Image from Hannah Markwig.
[63]Image from Hannah Markwig.

5.38 Definition. Σ_1, Σ_2 fans in $\mathbb{R}^n, \mathbb{R}^m$, $f : \mathrm{supp}(\Sigma_1) \to \mathrm{supp}(\Sigma_2)$ is called a map of fans if it is $\mathbb{Z}$-linear.
Construction/ Definition:
Let Σ_1 be equidumensional of $\dim d$ in $\mathbb{R}^n$, Σ_2 in $\mathbb{R}^m$, $f : \Sigma_1 \to \Sigma_2$ a map of fans. We construct the **push-forward/ image fan** $f_*(\Sigma_1)$ in Σ_2.
Idea: We want to use cones of the form $f(\sigma)$, $\sigma \subset \sigma'$ maximal in Σ_1 s.th. $f|_{\sigma'}$ is injective.
Problem: such images of cones could overlap.

5.39 Example. Consider $\sigma_1 = \{x \geq 0, z \geq 0\}$, $\sigma_2 = \{y \leq 0, z \leq 0\}$ (careful, these do not form a fan, since their intersection is not a face of both. However, this example is sufficient to get the idea of the problem. One would have to embed those cones into $\mathbb{R}^4$ and use "the fourth coordinate direction"to "separate"them in such a way, that one can build a fan).
$f : \mathbb{R}^3 \to \mathbb{R}^2 : (x, y, z) \mapsto (x, y)$

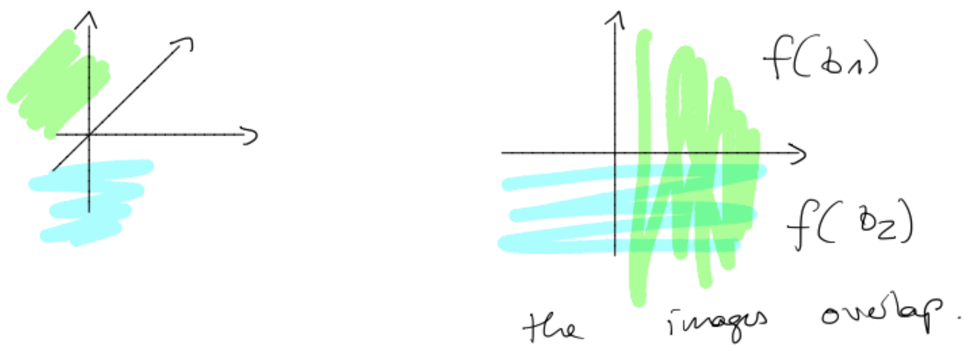
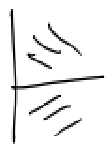
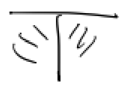


Abbildung 65: $f(\sigma_1)$ and $f(\sigma_2)$ [64]

But we consider fans only up to equivalence, so we can subdivide σ_1 and σ_2 s.th. the image of σ_1 is subdivided and that of σ_2, too.


(a) σ_1 subdivided [65]


(b) σ_2 subdivided [66]

In the following, we always assume that Σ_1 is suitably subdivided for f.
Then

$$f_*(\Sigma_1) := \{f(\sigma), \sigma \in \Sigma_1, \sigma \subset \sigma' \text{ maximal with } f|_{\sigma'} \text{ injective } \}$$

is a fan, equidimensional of $\dim d$. If Σ_1 is weighted, we also define weights for $f_*\Sigma_1$: For $\sigma' \in f_*\Sigma_1$ we let

$$\omega_{f_*\Sigma_1}(\sigma') = \sum_{\substack{\sigma \in \Sigma_1: \\ f(\sigma) = \sigma'}} \omega_{\Sigma_1}(\sigma) \cdot |\frac{\mathrm{span}(\sigma') \cap \mathbb{Z}^m}{f(\mathrm{span}\sigma \cap \mathbb{Z}^n)}|$$

5.40 Example. Σ_1 in $\mathbb{R}^2$, balanced since $\begin{pmatrix} -1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 0$

[64] Image from Hannah Markwig.
[66] Image from Hannah Markwig.

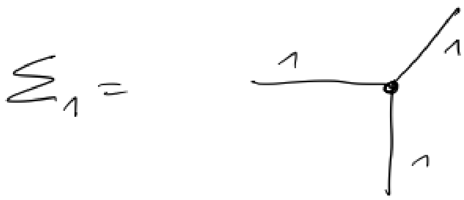


Abbildung 67: Σ_1 [67]

$f : \mathbb{R}^2 \to \mathbb{R} : (x, y) \mapsto x$

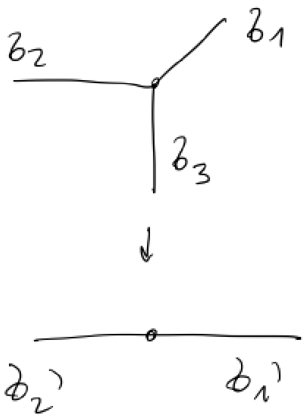


Abbildung 68: f [68]

$f|_{\sigma_3}$ is not injective, $f(\sigma_1) = \sigma_1'$, $f(\sigma_2) = \sigma_2'$. $\mathrm{span}\sigma_i' \cap \mathbb{Z} = \mathbb{Z}$, $\mathrm{span}\sigma_1 \cap \mathbb{Z}^2 = \left\langle \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\rangle_{\mathbb{Z}}$.
$f \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 1$ generates $\mathbb{Z} \Rightarrow$

$$|\frac{\mathrm{span}\sigma_1' \cap \mathbb{Z}}{f(\mathrm{span}\sigma_1 \cap \mathbb{Z}^2)}| = 1.$$

Analogously for σ_2, σ_2'.

Abbildung 69: $f_*\Sigma_1$ [69]

$f : \mathbb{R}^2 \to \mathbb{R} : (x, y) \mapsto x + y$

[67] Image from Hannah Markwig.
[68] Image from Hannah Markwig.
[69] Image from Hannah Markwig.
[70] Image from Hannah Markwig.

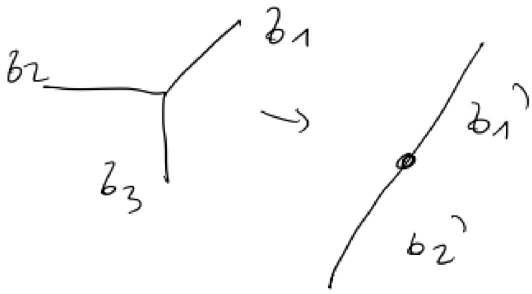


Abbildung 70: f [70]

$f(\sigma_1) = \sigma_1'$, $f(\sigma_2) = f(\sigma_3) = \sigma_2'$, $\text{span}(\sigma_i') \cap \mathbb{Z} = \mathbb{Z}$, $\text{span}(\sigma_1) \cap \mathbb{Z}^2 = \left\langle \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\rangle_{\mathbb{Z}}$,
$f\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 2 \Rightarrow f(\text{span}\sigma_1 \cap \mathbb{Z}^2) = 2\mathbb{Z}$

$$\Rightarrow |\frac{\text{span}\sigma_1' \cap \mathbb{Z}}{f(\text{span}\sigma_1 \cap \mathbb{Z}^2)}| = |\mathbb{Z}/2\mathbb{Z}| = 2$$

$\text{span}(\sigma_2) \cap \mathbb{Z}^2 = \left\langle \begin{pmatrix} -1 \\ 0 \end{pmatrix} \right\rangle_{\mathbb{Z}}$, $f\begin{pmatrix} -1 \\ 0 \end{pmatrix} = -1$ generates $\mathbb{Z} \Rightarrow$ Index 1
Analogously for σ_3.

$$\omega_{f_*\Sigma_1}(\sigma_2') = \omega_{\Sigma_1}(\sigma_2) \cdot \text{Index} + \omega_{\Sigma_1}(\sigma_3) \cdot \text{Index} = 1 \cdot 1 + 1 \cdot 1 = 2$$

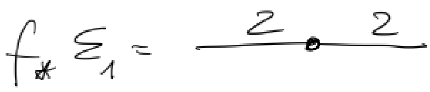


Abbildung 71: $f_*\Sigma_1$ [71]

5.41 Theorem. *Let $f : \Sigma_1 \to \Sigma_2$ be a map of fans, Σ_1 balanced, then $f_*\Sigma_1$ is also balanced.*

Beweis. We need to show the balancing condition at every codimension 1 cone $\tau' \in f_*\Sigma_1$. Let $\tau \in \Sigma_1$ of codim 1 with $f(\tau) = \tau'$. Around τ we have the balancing condition:

$$\sum_{\tau \subset \sigma} \omega_{\Sigma_1}(\sigma) \cdot u_{\sigma/\tau} = 0 \quad \text{in} \quad \mathbb{R}^n/\text{span}\tau$$

Apply f:

$$\sum_{\tau \subset \sigma} \omega_{\Sigma_1}(\sigma) \cdot f(u_{\sigma/\tau}) = 0 \quad \text{in} \quad \mathbb{R}^m/\text{span}\tau'$$

[71]Image from Hannah Markwig.

Let $\tau' \subset \sigma'$ and $\tau \subset \sigma$ s.th. $f(\sigma) = \sigma'$. The normal vectors $u_{\sigma'/\tau'}$ and $f(u_{\sigma/\tau})$ satisfy:

$$f(u_{\sigma/\tau}) = \left|\frac{\mathrm{span}\sigma' \cap \mathbb{Z}^m}{\mathrm{span}\tau' \cap \mathbb{Z}^m + \mathbb{Z} \cdot f(u_{\sigma/\tau})}\right| \cdot u_{\sigma'/\tau'}$$

if f is injective on σ and $f(u_{\sigma/\tau}) = 0$ else.
The following sequence is exact:

$$
\begin{aligned}
&0 \longrightarrow \frac{\mathrm{span}\tau' \cap \mathbb{Z}^m}{f(\mathrm{span}\tau \cap \mathbb{Z}^n)} \longrightarrow \frac{\mathrm{span}\sigma' \cap \mathbb{Z}^m}{f(\mathrm{span}\sigma \cap \mathbb{Z}^n)} \longrightarrow \frac{\mathrm{span}\sigma' \cap \mathbb{Z}^m}{\mathrm{span}\tau' \cap \mathbb{Z}^n + \mathbb{Z} \cdot f(u_{\sigma/\tau})} \longrightarrow 0 \\
\Rightarrow &\left|\frac{\mathrm{span}\tau' \cap \mathbb{Z}^m}{f(\mathrm{span}\tau \cap \mathbb{Z}^n)}\right| \cdot \left|\frac{\mathrm{span}\sigma' \cap \mathbb{Z}^m}{\mathrm{span}\tau' \cap \mathbb{Z}^n + \mathbb{Z} \cdot f(u_{\sigma/\tau})}\right| = \left|\frac{\mathrm{span}\sigma' \cap \mathbb{Z}^m}{f(\mathrm{span}\sigma \cap \mathbb{Z}^n)}\right| \\
\Rightarrow &\left|\frac{\mathrm{span}\sigma' \cap \mathbb{Z}^m}{\mathrm{span}\tau' \cap \mathbb{Z}^n + \mathbb{Z} \cdot f(u_{\sigma/\tau})}\right| = \frac{\left|\frac{\mathrm{span}\sigma' \cap \mathbb{Z}^m}{f(\mathrm{span}\sigma \cap \mathbb{Z}^n)}\right|}{\left|\frac{\mathrm{span}\tau' \cap \mathbb{Z}^m}{f(\mathrm{span}\tau \cap \mathbb{Z}^n)}\right|}
\end{aligned}
$$

Insert this into the equation above, the factor

$$\left|\frac{\mathrm{span}\tau' \cap \mathbb{Z}^m}{f(\mathrm{span}\tau \cap \mathbb{Z}^n)}\right|^{-1}$$

is the same for each summand and can therefore be taken out:

$$
\begin{aligned}
&\sum_{\tau \subset \sigma} \omega_{\Sigma_1}(\sigma) \cdot \left|\frac{\mathrm{span}\sigma' \cap \mathbb{Z}^m}{\mathrm{span}\tau' \cap \mathbb{Z}^n + \mathbb{Z} \cdot f(u_{\sigma/\tau})}\right| \cdot u_{\sigma'/\tau'} = 0 \quad \text{in} \quad \mathbb{R}^m/\mathrm{span}\tau' \\
\Rightarrow &\sum_{\tau \subset \sigma} \omega_{\Sigma_1}(\sigma) \cdot \left|\frac{\mathrm{span}\sigma' \cap \mathbb{Z}^m}{f(\mathrm{span}\sigma \cap \mathbb{Z}^n)}\right| \cdot u_{\sigma'/\tau'} = 0 \quad \text{in} \quad \mathbb{R}^m/\mathrm{span}\tau'
\end{aligned}
$$

Now we sum over all τ s.th. $f(\tau) = \tau'$: In $\mathbb{R}^m/\mathrm{span}\tau'$,

$$
\begin{aligned}
0 &= \sum_{\substack{\tau: \\ f(\tau)=\tau'}} \sum_{\tau \subset \sigma} \omega_{\Sigma_1}(\sigma) \cdot \left|\frac{\mathrm{span}\sigma' \cap \mathbb{Z}^m}{f(\mathrm{span}\sigma \cap \mathbb{Z}^n)}\right| \cdot u_{\sigma'/\tau'} \\
&= \sum_{\tau' \subset \sigma'} \left(\sum_{\substack{\sigma: \\ f(\sigma)=\sigma'}} \omega_{\Sigma_1}(\sigma) \cdot \left|\frac{\mathrm{span}\sigma' \cap \mathbb{Z}^m}{f(\mathrm{span}\sigma \cap \mathbb{Z}^n)}\right| \right) \cdot u_{\sigma'/\tau'} \\
&= \sum_{\tau' \subset \sigma'} \omega_{f_* \Sigma_1}(\sigma') \cdot u_{\sigma'/\tau'}
\end{aligned}
$$

$\square$

6 Regular subdivisions and the secondary fan

6.1 Definition. A **simplex** is a polytope of dim d with $d+1$ vertices

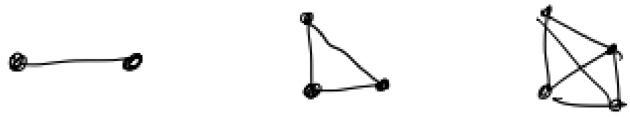


Abbildung 72: Simplices [72]

6.2 Definition. A **triangulation** of a polytope P is a subdivision into finitely many simplices s.th. the intersection of two such simplices is a face of both.

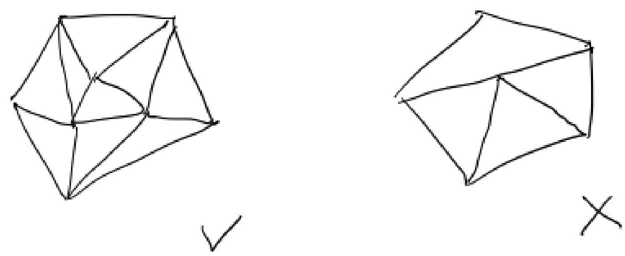


Abbildung 73: (Not a) triangulation [73]

For lattice polygons, we consider triangulations in the lattice, i.e. we require that the vertices of the simplices are in $P \cap \mathbb{Z}^n$. We do not require all points of $P \cap \mathbb{Z}^n$ to be vertices of the triangulation.

Abbildung 74: Triangulations on a lattice [74]

More generally, we can fix $A \subset P \cap \mathbb{Z}^n$ (or $A \subset P$ finite containing the vertices of P) and consider triangulations with vertices in A.
<u>Construction:</u> Let $P \subset \mathbb{R}^n$ be a fulldim lattice polytope. Let

$$\Psi : \underset{(P\cap\mathbb{Z}^n)}{A} \to \mathbb{R}$$

[72] Image from Hannah Markwig.
[73] Image from Hannah Markwig.
[74] Image from Hannah Markwig.

be a "**height function**". We project the upper faces (i.e. those with an outer normal vector whose last coordinate is positive) of

$$\operatorname{conv}\{(u,z) | z \leq \Psi(u) | u \in A, z \in \mathbb{R}\} \subset \mathbb{R}^n \times \mathbb{R} \quad \text{to } P.$$

6.3 Example. .

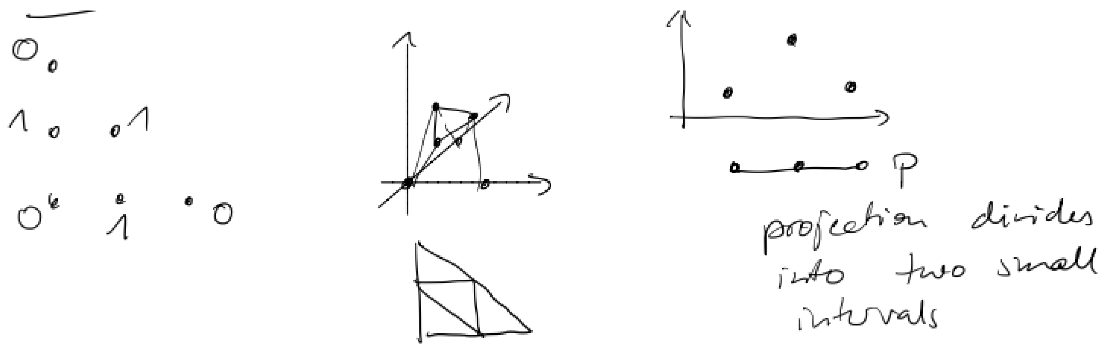


Abbildung 75: Projection of the upper faces [75]

For generic functions Ψ, the result is a triangulation: assume Q is a face which is no simplex, then Q has at least $n+2$ vertices, $v_1, \ldots, v_{n+2}$, and the points $\begin{pmatrix} v_1 \\ \Psi(v_1) \end{pmatrix}, \ldots, \begin{pmatrix} v_{n+2} \\ \Psi(v_{n+2}) \end{pmatrix}$ lie on an affine hyperplane in $\mathbb{R}^{n+1}$, i.e. they satisfy an equation

$$\left\langle m, \begin{pmatrix} v_i \\ \Psi(v_i) \end{pmatrix} \right\rangle = b$$

$$\Rightarrow \det \begin{pmatrix} v_1 & \ldots & \Psi(v_1) & b \\ \vdots & \vdots & \vdots & \vdots \\ \underbrace{v_{n+2}}_{n} & \ldots & \Psi(v_{n+2}) & b \end{pmatrix} = 0$$

$$\Rightarrow \det \begin{pmatrix} v_1 & \ldots & \Psi(v_1) & 1 \\ \vdots & \vdots & \vdots & \vdots \\ v_{n+2} & \ldots & \Psi(v_{n+2}) & 1 \end{pmatrix} = 0$$

only columns $\begin{pmatrix} \Psi(v_1) \\ \Psi(v_{n+2}) \end{pmatrix}$ which are contained in the linear span of $\begin{pmatrix} v_1 & 1 \\ \vdots & \vdots \\ v_{n+2} & 1 \end{pmatrix}$ $(n+1$ vectors in $\mathbb{R}^{n+2})$ do not produce a triangulation.

Abbildung 76: triangulation [76]

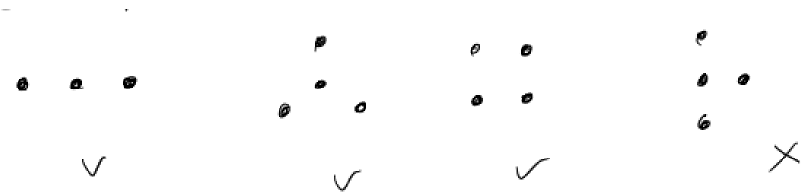


Abbildung 77: (no) circuit(s) [77]

6.4 Definition. A **circuit** Z is a set of points which are affinely dependent, s.th. every strict subset is affinely independent. (Simplex $\cup$ a point, see figure 77)

6.5 Lemma. *Let $\{v_1, \dots, v_k\}$ be a circuit. Then there exist coefficients $\lambda_1, \dots, \lambda_k$ which are unique up to scalar multiple s.th. $\sum \lambda_i v_i = 0$, $\sum \lambda_i = 0$, $\lambda_i \neq 0 \, \forall \, i$.*

Beweis. $v_1, \dots, v_k$ is affinely dependent
$\Rightarrow \exists \, (\lambda_1, \dots, \lambda_k) \neq (0, \dots, 0) : \sum \lambda_i v_i = 0, \sum \lambda_i = 0$.
We need to show that $\lambda_i \neq 0 \, \forall \, i$. Assume without restriction $\lambda_1 = 0$. Then

$$\sum_{l=2}^{k} \lambda_l v_l = 0 \quad \text{and} \quad \sum_{l=2}^{k} \lambda_l = 0$$

$\Rightarrow v_2, \dots, v_k$ is affinely dependent $\lightning$.
Finally, we need to show that the λ_i are unique up to scalar multiple.
Let $\sum \mu_i v_i = 0, \sum \mu_i = 0$. As before, all $\mu_i \neq 0$. We can scale both equations s.th. $\lambda_1 = \mu_1 = 1$. Then

$$\sum \lambda_i v_i - \sum \mu_i v_i = 0 - 0 = 0 = \sum_{i=2}^{k} (\lambda_i - \mu_i) v_i$$

and

$$\sum \lambda_i - \sum \mu_i = 0 - 0 = 0 = \sum_{i=2}^{k} (\lambda_i - \mu_i) \Rightarrow \lambda_i = \mu_i \, \forall \, i,$$

as $v_2, \dots, v_k$ are affinely independent. ☐

6.6 Definition. Let $Z = \{v_1, \dots, v_k\}$ be a circuit with $\sum \lambda_i v_i = 0$, $\sum \lambda_i = 0$ where the $\lambda_i \neq 0$ are unique up to scalar. Define

$$Z^+ = \{v_i | \lambda_i > 0\} \quad \text{and} \quad Z^- = \{v_i | \lambda_i < 0\}.$$

This is well-defined up to swapping.

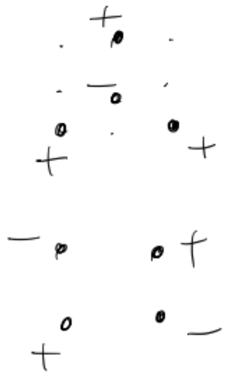
6.7 Example. To figure 78a: as $1 \cdot 0 + 1 \cdot 2 - 2 \cdot 1 = 0$, if we use coordinates $0, 1, 2$ for the points, or more generally, $1 \cdot p + 1 \cdot (p + 2) - 2 \cdot (p + 1) = 0$

To figure 78a (top): $1 \cdot \begin{pmatrix} 0 \\ 0 \end{pmatrix} + 1 \cdot \begin{pmatrix} 2 \\ 0 \end{pmatrix} + 2 \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} - 4 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 0$

To figure 78a (bottom): $1 \cdot \begin{pmatrix} 0 \\ 0 \end{pmatrix} + 1 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} - 1 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} - 1 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0$

[75] Image from Hannah Markwig.
[76] Image from Hannah Markwig.
[77] Image from Hannah Markwig.
[79] Image from Hannah Markwig.

(a) Z^+ and Z^- [78]



(b) Z^+ and Z^- [79]

6.8 Proposition. *Let Z be a circuit and $P = \text{conv}(Z)$. Then Z has precisely 2 triangulations with vertices in Z,*

$$T^+ = \{\text{conv}(Z \setminus \{w\}) | w \in Z^+\}$$
$$T^- = \{\text{conv}(Z \setminus \{w\}) | w \in Z^-\}$$

6.9 Example. .

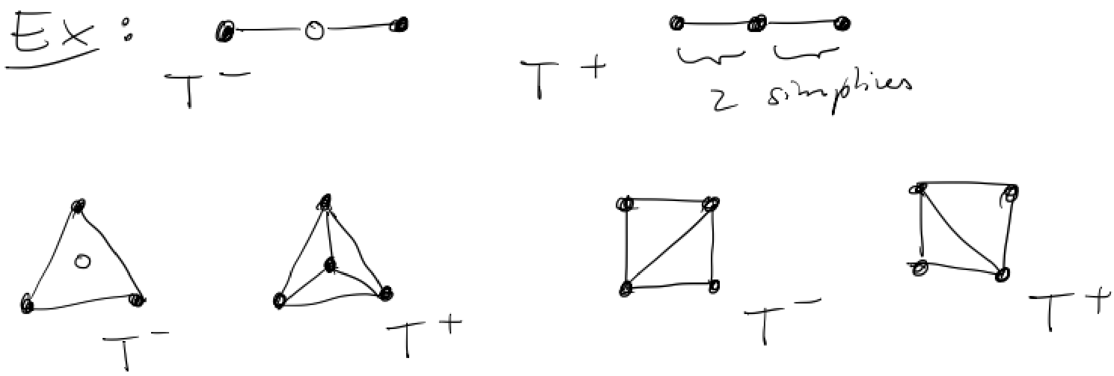


Abbildung 79: T^+ and T^- [80]

Beweis. Let $x \in P$, $x = \sum_{\omega \in Z} \mu_\omega \omega$ with $\mu_\omega \geq 0, \sum \mu_\omega = 1$.
This expression for x is not unique, as we can add multiples of $\sum \lambda_\omega \omega = 0$ while keeping the coefficients nonnegative, since the sum of the coefficients will stay 1 as $\sum \lambda_\omega = 0$.
Let $\omega' \in Z^+$ s.th. $\mu_\omega / \lambda_\omega$ is minimal. Then

$$\begin{aligned} X &= \sum_{\omega \in Z} \mu_\omega \omega - \frac{\mu_{\omega'}}{\lambda_{\omega'}} \sum_{\omega \in Z} \lambda_\omega \omega \\ &= \sum_{\omega \in Z} (\mu_\omega - \frac{\mu_{\omega'} \lambda_\omega}{\lambda_{\omega'}}) \omega \\ &= \sum_{\omega \neq \omega'} (\mu_\omega - \frac{\mu_{\omega'} \lambda_\omega}{\lambda_{\omega'}}) \omega \end{aligned}$$

is a convex combination, since

[80] Image from Hannah Markwig.

- if $\omega \in Z^+$:

$$\frac{\mu_\omega}{\lambda_\omega} \geq \frac{\mu_{\omega'}}{\lambda_{\omega'}} \overset{\lambda_\omega > 0}{\Longleftrightarrow} \mu_\omega \geq \frac{\mu_{\omega'}\lambda_\omega}{\lambda_{\omega'}} \Longleftrightarrow \mu_\omega - \frac{\mu_{\omega'}\lambda_\omega}{\lambda_{\omega'}} \geq 0$$

- if $\omega \in Z^-$:

$$\underbrace{\frac{\mu_{\omega'}}{\lambda_{\omega'}}}_{>0} \cdot \underbrace{\lambda_\omega}_{<0} < 0 \Rightarrow \mu_\omega - \frac{\mu_{\omega'}\lambda_\omega}{\lambda_{\omega'}} > \mu_\omega \geq 0$$

$\Rightarrow$ $x \in \mathrm{Conv}(Z \setminus \{w'\})$ where w' is chosen as above
$\Rightarrow$ $P = \bigcup\limits_{w' \in Z^+} \mathrm{Conv}(Z \setminus \{w'\})$.

If x is contained in two or more of the $\mathrm{Conv}(Z \setminus \{w'\})$, the minimum of the $\frac{\mu_\omega}{\lambda_\omega}$ for $\omega \in Z^+$ is taken multiple times (say for w', w''), and $x \in \mathrm{Conv}(Z \setminus \{\omega', \omega''\})$ which is a face of both, hence

$$T^+ = \{\mathrm{Conv}(Z \setminus \{\omega'\}) |\, \omega' \in Z^+\}$$

is a triangulation. Analogously, we can see that T^- is a triangulation.
Assume there existed more triangulations than T^+ and T^-. One such further triangulation T would have to combine simplices in T^+ with simplices in T^-, as these are all simplices that exist with vertices in Z. Assume $\mathrm{Conv}(Z \setminus \{\omega'\})$ and $\mathrm{Conv}(Z \setminus \{\omega''\})$ are in T, with $\omega' \in Z^+$, $\omega'' \in Z^-$. A point $x \in \mathrm{Conv}(Z \setminus \{\omega'\})$ is, as before, given as $X = \sum \mu_\omega \omega$ s.th. $\frac{\mu_\omega}{\lambda_\omega}$ is minimal at ω' for $\omega \in Z^+$. X is in the relative interior of

$$\mathrm{Conv}(Z \setminus \{\omega'\}) \Leftrightarrow \mu_\omega - \frac{\mu_{\omega'}\lambda_\omega}{\lambda_{\omega'}} > 0 \;\forall\, \omega \neq \omega' \Leftrightarrow \text{ the minimum is unique at } \omega'.$$

Analogously: a point is in the interior of $\mathrm{Conv}(Z \setminus \{\omega''\})$ if $X = \sum \mu_\omega \omega$, $\frac{\mu_\omega}{\lambda_\omega}$ is maximal at ω'' for $\omega \in Z^-$, and the maximum is unique.
Given ω' and ω'' in Z^+ resp. Z^-, one can find coeff. μ_ω satisfying both conditions $\Rightarrow$ x is in the interior of both simplices $\Rightarrow$ T is not a triangulation.

Abbildung 80: not a triangulation [81]

$\square$

6.10 Definition. Let T be a triangulation of P. $g : P \to \mathbb{R}$ is called **T-piecewise linear**, if g is affine-linear on each simplex of T
$\Leftrightarrow$ the graph of g is a "roof" whose corners are (at most) at the facets of the maximal simplices of T.
$g : P \to \mathbb{R}$ is called **concave**:$\Leftrightarrow$

$$\forall\, x, y \in P,\, 0 \leq t \leq 1 :\, g(tx + (1-t)y) \geq tg(x) + (1-t)g(y)$$

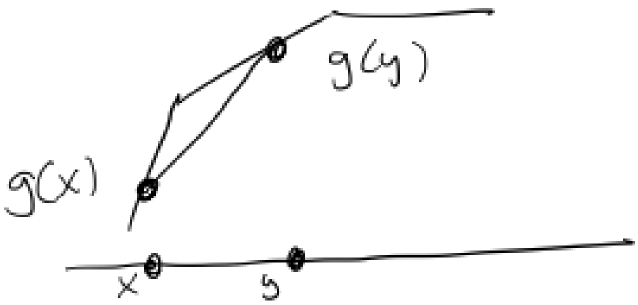


Abbildung 81: line segments [82]

$\Leftrightarrow$ line segments of points on the roof are below the roof (see figure 81).

A **domain of linearity** of a piecewise linear $g : P \to \mathbb{R}$ is a maximal $U \subset P$ s.th. $g|_U$ is linear.

6.11 Definition. A triangulation T is **regular** $:\Leftrightarrow \exists$ concave T-piecewise linear $g : P \to \mathbb{R}$ whose domains of linearity are exactly the minimal simplices of T.

6.12 Example. A triangulation which is not regular (Exercise):

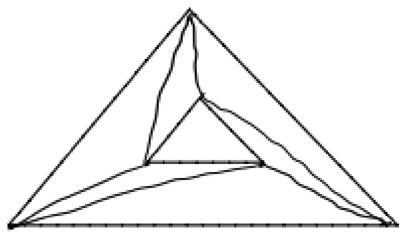


Abbildung 82: A not regular triangulation [83]

6.13 Remark. Let P be a polytope, $A \subset P$ finite, $\Psi : A \to \mathbb{R}$ a **height function**, T a triangulation of P with vertices in A.
Then there exists a unique T-piecewise linear function $g_{\Psi,T} : P \to \mathbb{R}$ s.th.
$g_{\Psi,T}(\omega) = \Psi(\omega) \,\forall\, \omega \in A$ which are vertices of a simplex of T, linearly continued, given by affine-linear continuation in each simplex. Values of Ψ at points which are not vertices of a simplex of T do not have any effect on $g_{\Psi,T}$.

6.14 Example. See figure 83.

6.15 Definition. Let T be a triangulation with vertices in A of P. Define $C(T) \subset \mathbb{R}^{\#A}$ by

$$
\begin{aligned}
C(T) = \{ & \Psi : A \to \mathbb{R} \text{ satisfying} \\
& - g_{\psi,T} : P \to \mathbb{R} \text{ is concave} \\
& - \forall\, \omega \in A \text{ which are not a vertex of a simplex of } T : g_{\Psi,T}(\omega) \geq \Psi(\omega) \}
\end{aligned}
$$

$C(T)$ is called the **secondary cone** of T.

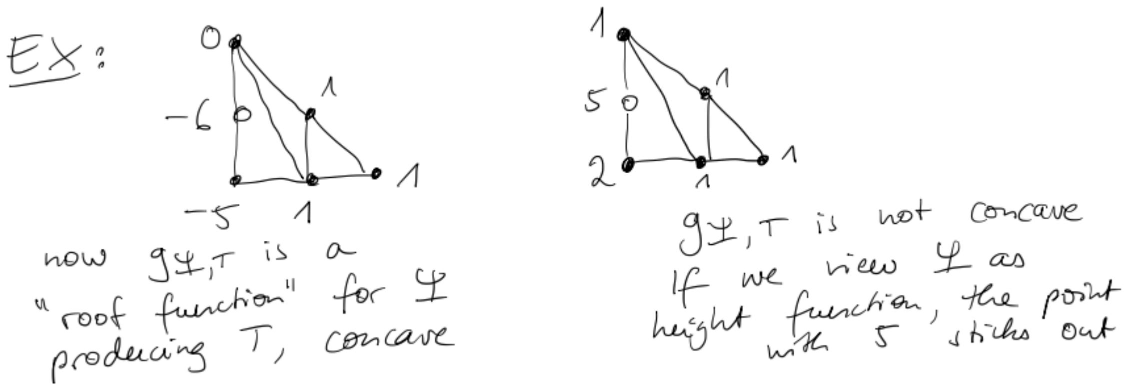


Abbildung 83: A concave and a not concave $g_{\Psi,T}$ [84]

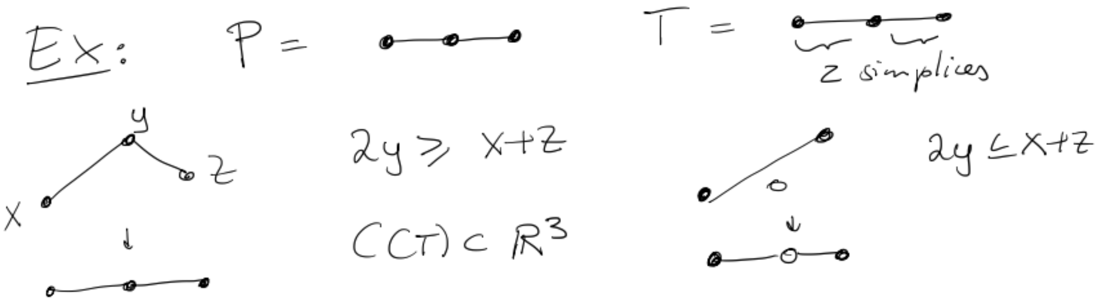


Abbildung 84: P and T [85]

6.16 Example. See figure 84.

There are two triangulations. Their two secondary cones subdivide $\mathbb{R}^3$ via the hyperplane $2y = x + z$.

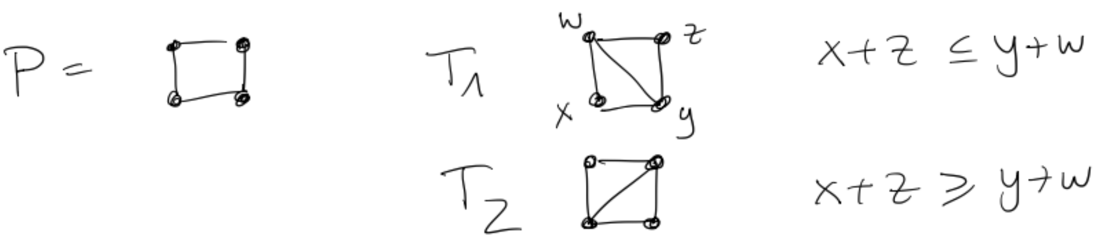


Abbildung 85: P, T_1 and T_2 [86]

$\mathbb{R}^4$ subdivided by the hyperplane $x + z = y + w$ yields the two secondary cones.
In the hyperplane separating the two cones, $x + z = y + w$, we obtain a regular subdivision which is not a triangulation by projecting the "roof function":

[81] Image from Hannah Markwig.
[82] Image from Hannah Markwig.
[83] Image from Hannah Markwig.
[84] Image from Hannah Markwig.
[85] Image from Hannah Markwig.
[86] Image from Hannah Markwig.
[87] Image from Hannah Markwig.

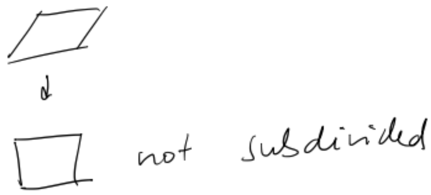


Abbildung 86: not subdivided [87]

The definition of secondary cone is build such that such more general subdivisions can be part, too:
the domains of linearity of $g_{\Psi,T}$ do not have to be the simplices of T, they can be larger (unions of simplices of T, if the two affine-linear continuations of 2 neighbouring simplices happen to fit together.

6.17 Lemma. *(1) T regular $\Leftrightarrow C(T)^\circ \neq \emptyset$*

(2) $\Psi \in C(T)^\circ \Leftrightarrow$ Using Ψ as height function and projection the upper faces of the convex hull of its graph yields T

(3) $C(T)$ is a cone

Beweis. (1) T regular $\Leftrightarrow \exists\, g_{\Psi,T}$ for $\Psi \in C(T)$ whose domains of linearity are exactly the simplices of T $\Leftrightarrow \exists\, g_{\Psi,T}$ for $\Psi \in C(T)$ and it is not on the boundary, i.e. satisfying additional relations corresponding to domains of linearity fitting together
$\Leftrightarrow \exists\, g_{\Psi,T}$ for $\Psi \in C(T)^\circ$.

(2) Given $\Psi \in C(T)^\circ$, it yields $g_{\Psi,T}$ which is concave by definition, $g_{\Psi,T}(w) = \Psi(w)\,\forall\, w \in A$ which are vertices of simplices of T and $g_{\Psi,T} \geq \Psi(w)$ for all other w. Thus, the graph of $g_{\Psi,T}$ is the convex hull of the graph of Ψ, and projecting its upper faces yields its domains of linearity. As we are not in the boundary, these are exactly the simplices of T.
Vice versa, if we obtain T from the height function Ψ, we get $g_{\Psi,T}$ from its graph which we define to be the convex hull of the graph of Ψ. Changing the heights slightly will not change the projection, because T is a triangulation, so we are in $C(T)^\circ$.

(3) Scaling a "roof function" or adding two with the same corner locus (Knickstelle) does not change this corner locus. The boundary of $C(T)$ is given by hyperplanes that describe the affine linear functions of neighbouring simplices to fit to each other, or interior points of a simplex "hitting the roof "and sticking out. Thus $C(T)$ is a polyhedral cone.

□

6.18 Theorem. *The cones $C(T)$ for all triangulations T of P with vertices in A together with their faces form a complete fan in $\mathbb{R}^{\#A}$, the **secondary fan** of (P, A).*

6.19 Example. $\mathbb{R}^3$ divided by the hyperplane $2y = x + y$ is the secondary fan (figure 87a). (Two triangulations T^+, T^-.)
$\mathbb{R}^4$ divided by $x + z = y + w$, see figure 87b.

[89] Image from Hannah Markwig.

(a) in $\mathbb{R}^3$ [88] (b) in $\mathbb{R}^4$ [89]

6.20 Remark. (1) The secondary fan is not defined as the dual fan of a polytope, but it is dual to a polytope, the secondary polytope (later).

(2) The secondary fan is not pointed: $(1, \dots, 1)$ is in the lineality space, since shifting a roof function does not change the induced triangulation. Also, if $A = \{v_1, \dots, v_k\}$ then $\forall\, i = 1, \dots, n$, $(v_{1_i}, \dots, v_{k_i}) \in \mathbb{R}^{\#A}$ is in the lineality space.
Adding such a vector amounts to "turning" the roof function and does not change the induced triangulation.

6.21 Example. $\varPsi \in C(T)^\circ$ has a triangle/ triangulation between the 1s.

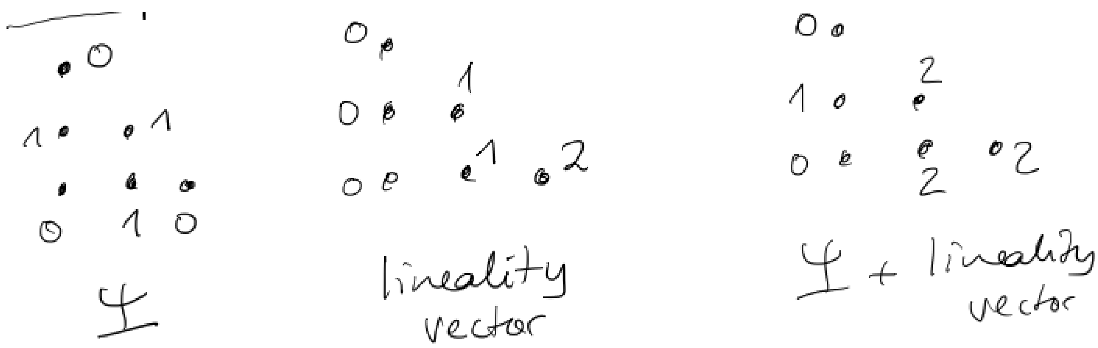


Abbildung 88: $\varPsi$ plus the lineality vector [90]

Beweis. Let $\varPsi \in \mathbb{R}^{\#A}$, then projecting the upper faces of the convex hull of the graph of $\varPsi$ yields a subdivision. If it is a triangulation T, then $\varPsi \in C(T)$. If it is not a triangulation, we can refine it to a triangulation T, and then $\varPsi$ is in (the boundary of) $C(T)$. Hence, the secondary fan is complete.
To see that it is a fan at all, note first that faces are contained by definition.
We have to show that the intersection $C(T) \cap C(T')$ is a face of both. Let $\varPsi \in C(T) \cap C(T')$. $g_{\psi,T}$ and $g_{\varPsi,T'}$ are both concave. If $\sigma \in T$, $\sigma' \in T'$ are maximal simplices which intersect in the interior, $g_{\varPsi,T}$ and $g_{\varPsi,T'}$ must be defined by the same affine-linear function on $\sigma \cup \sigma'$. This yields a linear condition on $\varPsi \Rightarrow C(T) \cap C(T')$ is the intersection of $C(T)$ with a linear subspace given by these conditions, and the same holds for $C(T')$. Thus it is a face of both. $\square$

Next we'll show that the secondary fan is dual to a polytope.

6.22 Definition. Let P be a fulldimensional polytope, $A \subset P$ finite containing the vertices. Let T be a triangulation with vertices in A. The characteristic function of T is

$$\varphi_T : A \to \mathbb{R} : \omega \mapsto \sum_{\substack{\sigma \text{ max simplex:} \\ \omega \text{ is vertex of } \sigma}} \operatorname{vol}(\sigma)$$

[90] Image from Hannah Markwig.

The **secondary polytope** of (P, A) is

$$\mathrm{Conv}(\varphi_T(\omega) | T \text{ triangulation}) \subset \mathbb{R}^{\#A}$$

6.23 Remark. We identify $\mathbb{R}^{\#A} \cong (\mathbb{R}^{\#A})^\vee$ by using the Euclidean scalar product

$$\langle \Psi, \varphi \rangle = \sum_{\omega \in A} \Psi(\omega) \cdot \varphi(\omega).$$

6.24 Example. The secondary polytope is the interval connecting these two points. This line segment is orthogonal to the hyperplane $y + z = x + w$, thus the secondary fan is the normal fan.

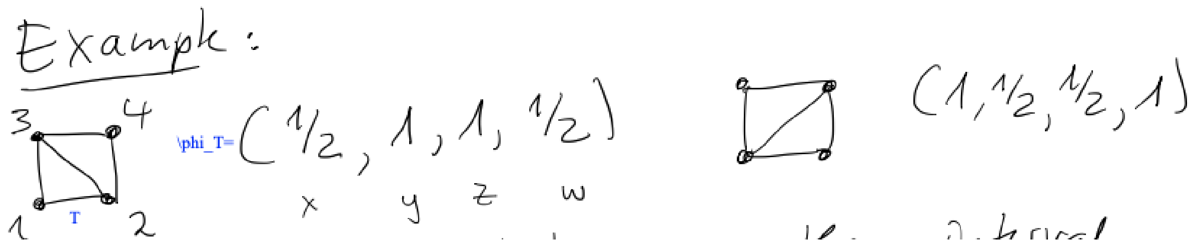


Abbildung 89: polytopes [91]

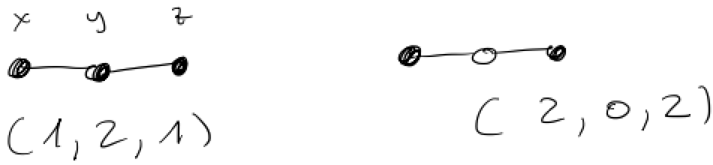


Abbildung 90: secondary polytopes [92]

The secondary polytope is the line segment connecting these two vertices. It is orthogonal to the hyperplane $2y = x + z$ in $\mathbb{R}^3$.

6.25 Lemma. *$\forall$ triangulations T and $\Psi : A \to \mathbb{R}$ we have*

$$\langle \Psi, \varphi_T \rangle = (n+1) \int_P g_{\Psi,T}(x) dx$$

Beweis. The integral of an affine-linear function over a simplex σ is $\mathrm{vol}(\sigma)\cdot$ the arithmetic mean of the values at the vertices

[91]Image from Hannah Markwig.
[92]Image from Hannah Markwig.
[93]Image from Hannah Markwig.

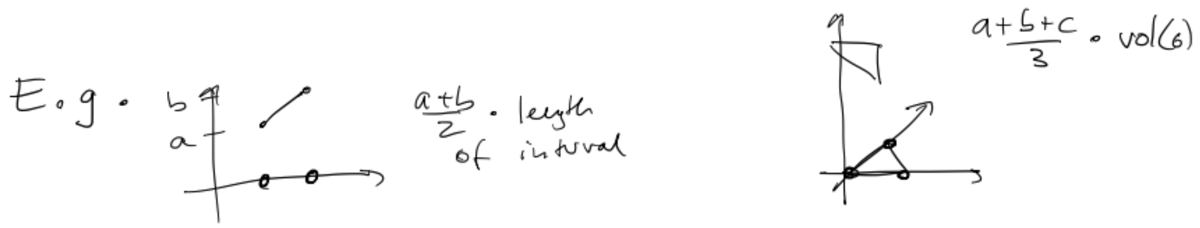


Abbildung 91: example [93]

Let $\sigma_1, \ldots, \sigma_l$ be the maximal simplices of T, then

$$\begin{aligned}
(n+1)\int_P g_{\psi,T}(x)dx &= (n+1)\cdot\left(\int_{\sigma_1} g_{\Psi,T}(x)dx + \ldots + \int_{\sigma_l} g_{\Psi,T}(x)dx\right)\\
&= \left(\sum_{\substack{\omega \text{ vertices}\\ \text{of } \sigma_1}} g_{\Psi,T}(\omega)\right)\cdot \mathrm{vol}(\sigma_1) + \ldots + \left(\sum_{\substack{\omega \text{ vertices}\\ \text{of } \sigma_l}} g_{\Psi,T}(\omega)\right)\cdot \mathrm{vol}(\sigma_l)\\
&= \sum_{\omega} g_{\Psi,T}(\omega)\cdot\left(\sum_{\substack{\omega \text{ vertex}\\ \text{of } \sigma \text{ in } T}} \mathrm{vol}(\sigma)\right)\\
&= \langle \Psi, \varphi_T\rangle
\end{aligned}$$

□

Notation: Let $\Psi \in \mathbb{R}^{\#A}$. We consider the convex hull of the graph of Ψ, the "roof" of

$$G_\Psi = \mathrm{Conv}((\omega, y) | y \leq \Psi(\omega), \omega \in A, y \in \mathbb{R})$$

The upper faces (the roof) is the graph of a piecewise linear function $g_\Psi : P \to \mathbb{R}$, $g_\Psi(x) = \max\{y : (x,y) \in G_\Psi\}$

6.26 Lemma. *Let $\Psi \in \mathbb{R}^{\#A}$.*

(1) $\forall$ triangulations T with vertices on A $g_\Psi(x) \geq g_{\Psi,T}(x) \,\forall\, x \in P$

(2)

$$\max_{\varphi\in\mathrm{Secpoly}(P)} \langle \Psi, \varphi\rangle = (n+1)\int_P g_\Psi(x)dx$$

Beweis. (1) Consider a maximal simplex σ of T. $g_{\Psi,T}$ is affine-linear in σ and $g_\Psi(\omega) \geq \Psi(\omega) = g_{\Psi,T}(\omega) \,\forall\, \omega$ vertices of σ. Then the inequality holds for all points $x \in \sigma$.

(2)

$$\max_{\varphi} \langle \Psi, \varphi\rangle = \max_{\varphi_T} \langle \Psi, \varphi_T\rangle,$$

since the secondary polytope is the convex hull of the φ_T, and the maximum of a linear functional is always also taken at a vertex (the vertices are a priori among the φ_T, we will see later: they are the φ_T).

$$\begin{aligned}\max_{\varphi_T}\langle \Psi, \varphi_T\rangle &= \max_{\varphi_T}(n+1)\int_P g_{\Psi,T}(x)dx \\ &\overset{1)}{\leq} (n+1)\int_P g_\Psi(x)dx\end{aligned}$$

Equality follows if we find T s.th.

$$\int_P g_{\Psi,T}(x)dx = \int_P g_\Psi(x)dx.$$

We get this by finding T s.th. $g_{\Psi,T}(x) = g_\Psi(x)\,\forall\, x \in P$. Projecting the upper faces of the graph of g_Ψ, we obtain a subdivision of P, but not necessarily a triangulation. If we choose Ψ' in a small neighbourhood of Ψ, we obtain a triangulation T refining the subdivision. For this triangulation, we have $g_{\Psi,T} = g_\Psi$, since g_Ψ is affin-linear on each simplex of T as it is so already on the bigger cells of the less fine subdivison.

□

6.27 Theorem. *The secondary fan is the normal fan of the secondary polytope. The cone $C(T)$ corresponds to the vertex φ_T. In particular, all φ_T are vertices of the secondary polytope.* Secfan $= N$(Secpoly), $C(T) \cong \varphi_T$.

Beweis. Let $\Psi \in C(T)^\circ$, then the projection of the graph of g_Ψ yields the triangulation T and we have

$$\max_{\varphi}\langle \Psi, \varphi\rangle = (n+1)\int_P g_\Psi(x)dx = (n+1)\int_P g_{\Psi,T}(x)dx = \langle \Psi, \varphi_T\rangle$$

$\Rightarrow \Psi$ takes its maximum on $\varphi_T \Rightarrow \Psi$ is in the cone of the normal fan of the secondary polytope corresponding to φ_T.
Let Ψ be in the cone of the normal fan of the secondary polytope corresponding to φ_T $\Leftrightarrow \Psi$ takes its maximum at φ_T

$$\begin{aligned}\Rightarrow \max_{\varphi\in \text{SecPoly}}\langle \Psi, \varphi\rangle = \langle \Psi, \varphi_T\rangle &= (n+1)\int_P g_{\Psi,T}(x)dx \\ &\leq (n+1)\int_P g_\Psi(x)dx = \max_{\varphi}\langle \Psi, \varphi\rangle\end{aligned}$$

$\Rightarrow g_\Psi = g_{\Psi,T}$ is concave, and for every $\omega \in A$, $g_{\Psi,T}(\omega) = g_\Psi(\omega) \geq \Psi(\omega)$, in particular for ω which are not vertices of simplices of $T \Rightarrow g_{\Psi,T}$ meets the requirements and yields $\Psi \in C(T)$. □

6.28 Corollary. *The secondary polytope has dimension $\#A - n - 1$.*

Beweis. Since the lineality space of the secondary fan has dimension $n+1$, the polytope has dim $\#A - n - 1$. □

6.29 Example. See figure 92.

Next, we study further faces of the secondary polytope, resp. cones of higher codim of the secondary fan.

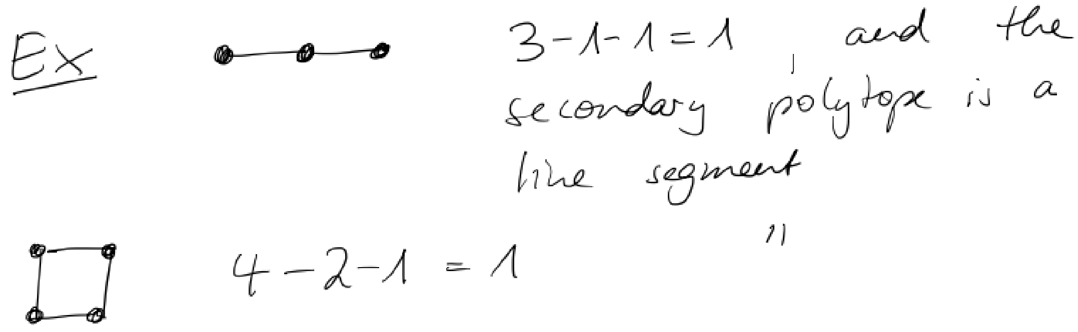


Abbildung 92: $3-1-1$ and $4-2-1$ [94]

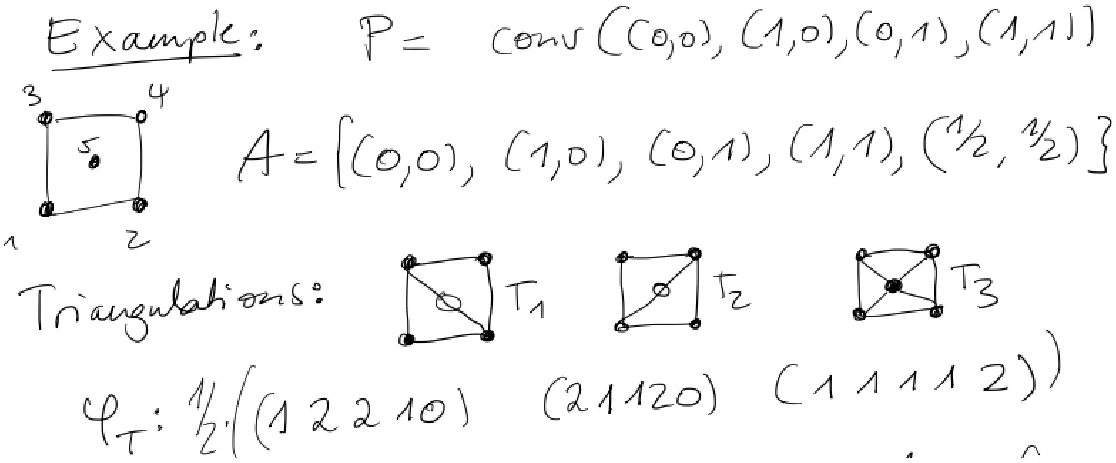


Abbildung 93: A and its triangulations [95]

6.30 Example. See figure 93.

The lineality space of the secondary fan is the rowspace of

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 & 1 & \frac{1}{2} \end{pmatrix}$$

The three points φ_T satisfy the equations

$$\begin{aligned} x_1 + \ldots + x_5 &= 3, \\ x_2 + x_4 + \frac{1}{2}x_5 &= \frac{3}{2}, \\ x_3 + x_4 + \frac{1}{2}x_5 &= \frac{3}{2} \end{aligned}$$

$\Rightarrow$ they lie in an affine space orthogonal to the lineality space. We consider the orthogonal complement of the lineality space, it is given by the two basis vectors

$$\begin{pmatrix} -1 & 0 & 0 & -1 & 2 \end{pmatrix}, \begin{pmatrix} 0 & -1 & -1 & 0 & 2 \end{pmatrix}.$$

[94]Image from Hannah Markwig.
[95]Image from Hannah Markwig.

We write the 3 points φ_T in the basis of $\mathbb{R}^5$ given by the 3 lineality vectors and the two in the orthogonal complement: (multiply both sides with $\frac{1}{2}$)

$$\begin{pmatrix}1\\2\\2\\1\\0\end{pmatrix} = (-\frac{4}{5})\cdot\begin{pmatrix}0\\-1\\-1\\0\\2\end{pmatrix} + (\frac{1}{5})\cdot\begin{pmatrix}-1\\0\\0\\-1\\2\end{pmatrix} + 0\cdot\begin{pmatrix}0\\0\\1\\1\\\frac{1}{2}\end{pmatrix} + 0\cdot\begin{pmatrix}0\\1\\0\\1\\\frac{1}{2}\end{pmatrix} + (\frac{6}{5})\cdot\begin{pmatrix}1\\1\\1\\1\\1\end{pmatrix}$$

$$\begin{pmatrix}2\\1\\1\\2\\0\end{pmatrix} = (\frac{1}{5})\cdot\begin{pmatrix}0\\-1\\-1\\0\\2\end{pmatrix} + (-\frac{4}{5})\cdot\begin{pmatrix}-1\\0\\0\\-1\\2\end{pmatrix} + 0\cdot\begin{pmatrix}0\\0\\1\\1\\\frac{1}{2}\end{pmatrix} + 0\cdot\begin{pmatrix}0\\1\\0\\1\\\frac{1}{2}\end{pmatrix} + (\frac{6}{5})\cdot\begin{pmatrix}1\\1\\1\\1\\1\end{pmatrix}$$

$$\begin{pmatrix}1\\1\\1\\1\\2\end{pmatrix} = (\frac{1}{5})\cdot\begin{pmatrix}0\\-1\\-1\\0\\2\end{pmatrix} + (\frac{1}{5})\cdot\begin{pmatrix}-1\\0\\0\\-1\\2\end{pmatrix} + 0\cdot\begin{pmatrix}0\\0\\1\\1\\\frac{1}{2}\end{pmatrix} + 0\cdot\begin{pmatrix}0\\1\\0\\1\\\frac{1}{2}\end{pmatrix} + (\frac{6}{5})\cdot\begin{pmatrix}1\\1\\1\\1\\1\end{pmatrix}$$

In the orthogonal complement of the lineality space, the three points have coordinates $\frac{1}{2}\begin{pmatrix}-\frac{4}{5}, & \frac{1}{5}\end{pmatrix}, \frac{1}{2}\begin{pmatrix}\frac{1}{5}, & -\frac{4}{5}\end{pmatrix}, \frac{1}{2}\begin{pmatrix}\frac{1}{5}, & \frac{1}{5}\end{pmatrix}$.

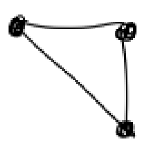


Abbildung 94: secondary polytope [96]

We can use this computation to convince ourselves that the secondary polytope is a triangle as expected (3 vertices, dim $5-2-1=2$), but we cannot compute the normal fan directly, for that we would have needed an orthonormal basis, which we'd rather avoid.
But we can compute the normal fan directly in the affine plane in $\mathbb{R}^5$ (figure 95)

$$v_1 = \text{vector orthogonal to } \underbrace{\frac{1}{2}(\begin{pmatrix}1 & 2 & 2 & 1 & 0\end{pmatrix} - \begin{pmatrix}2 & 1 & 1 & 2 & 0\end{pmatrix})}_{\frac{1}{2}\begin{pmatrix}-1 & 1 & 1 & -1 & 0\end{pmatrix}}$$

and to the lineality vectors, i.e. vector in the kernel of

$$\begin{pmatrix}1 & 1 & 1 & 1 & 1\\ 0 & 1 & 0 & 1 & \frac{1}{2}\\ 0 & 0 & 1 & 1 & \frac{1}{2}\\ -1 & 1 & 1 & -1 & 0\end{pmatrix}$$

[96] Image from Hannah Markwig.
[97] Image from Hannah Markwig.

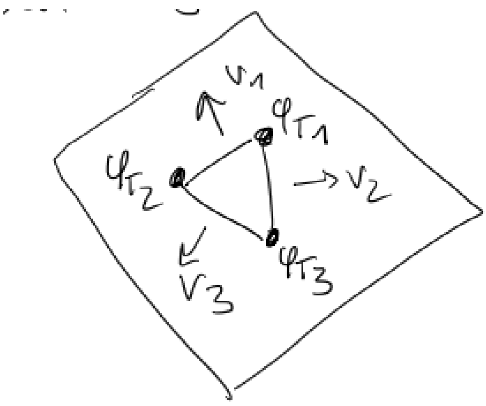


Abbildung 95: normal fan [97]

$v_1 = \begin{pmatrix} 1, & 1, & 1, & 1, & -4 \end{pmatrix}$, analogously
$v_2 = \begin{pmatrix} 2, & -3, & -3, & 2, & 2 \end{pmatrix}$, $v_3 = \begin{pmatrix} -3, & 2, & 2, & -3, & 2 \end{pmatrix}$
The secondary fan is:

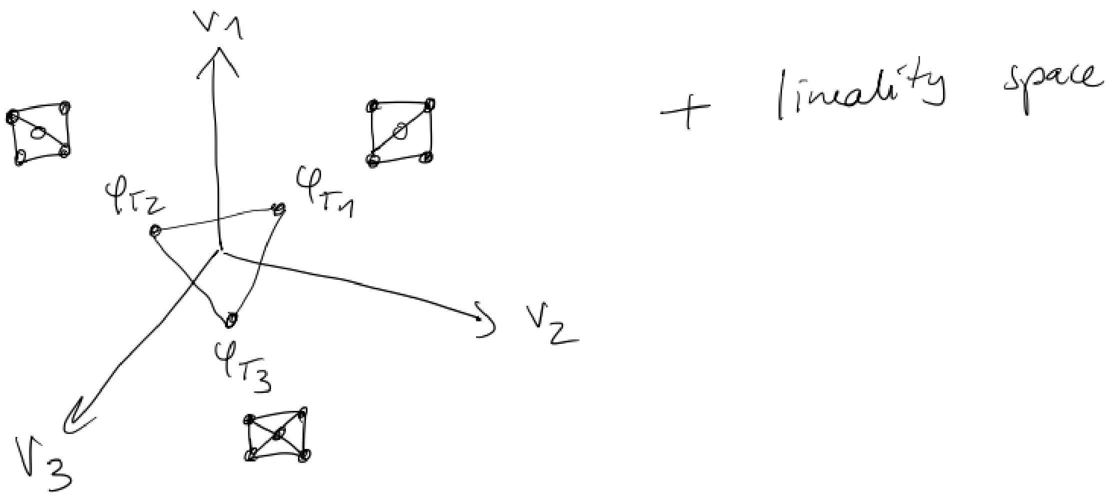
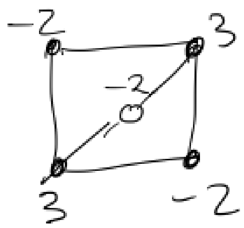


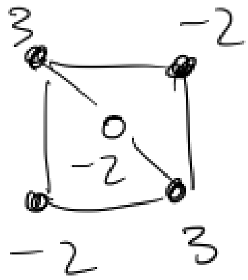
Abbildung 96: secondary fan [98]

We check the triangulations:
Pick a point in $\mathrm{Cone}(v_1, v_2)^\circ$, e.g.
$v_1 + v_2 = \begin{pmatrix} 3, & -2, & -2, & 3, & -2 \end{pmatrix}$ (figure 97a)
$v_1 + v_3 = \begin{pmatrix} -2, & 3, & 3, & -2, & -2 \end{pmatrix}$ (figure 97b)
$v_2 + v_3 = \begin{pmatrix} -1, & -1, & -1, & -1, & 4 \end{pmatrix}$ (figure 97c)
Let us compute the subdivisions for the rays in figure 98.

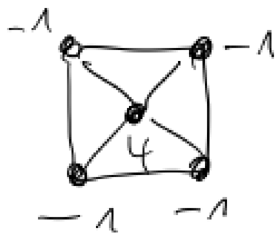
[98]Image from Hannah Markwig.
[102]Image from Hannah Markwig.



(a) T_1 [99]



(b) T_2 [100]



(c) T_3 [101]

6.31 Definition. (1) A **marked polytope** is a pair (Q, A) s.th. $A \subset Q$ is finite and contains the vertices of Q.

(2) Let (Q, A) be a marked polytope, $\dim Q = n$. A **marked subdivision** of (Q, A) is a set $\{(Q_i, A_i)_{i \in I}\}$ s.th.

a) $A_i \subset A \,\forall\, i, \dim Q_i = n$

b) $Q_i \cap Q_j$ is a face of Q_i and Q_j

c) $A_i \cap (Q_i \cap Q_j) = A_j \cap (Q_i \cap Q_j)$

d) $\bigcup Q_i = Q$

6.32 Example. (1) A triangulation can be viewed as a marked subdivision by marking the vertices of each simplex to be in A_i.

(2) Condition c) implies that we can draw pictures as in figure 99.

6.33 Remark. It is not required that $\bigcup A_i = A$.

6.34 Definition. Let S, S' be marked subdivisions $S = \{(Q_i, A_i)\}$, $S' = \{(Q'_j, A'_j)\}$ of the same marked polytope. We say S **refines** S' if $\forall\, j$ the set $\{(Q_i, A_i)$ s.th. $Q_i \subset Q'_j\}$ is a marked subdivision of (Q'_j, A'_j).
The set of marked subdivisions is a partially ordered set. Triangulations are minimal elements, maximal is $\{(Q, A)\}$.

[103]Image from Hannah Markwig.

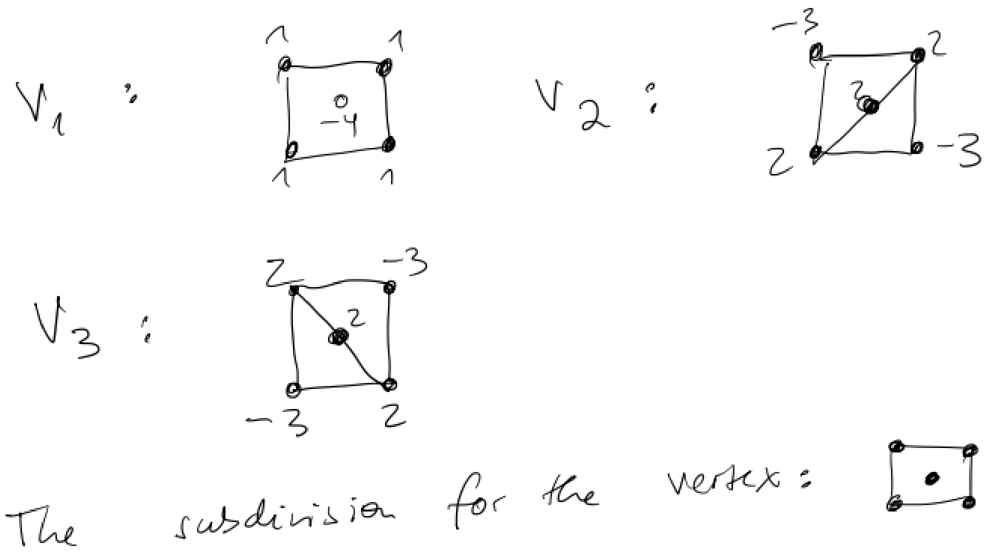


Abbildung 98: subdivisions [102]

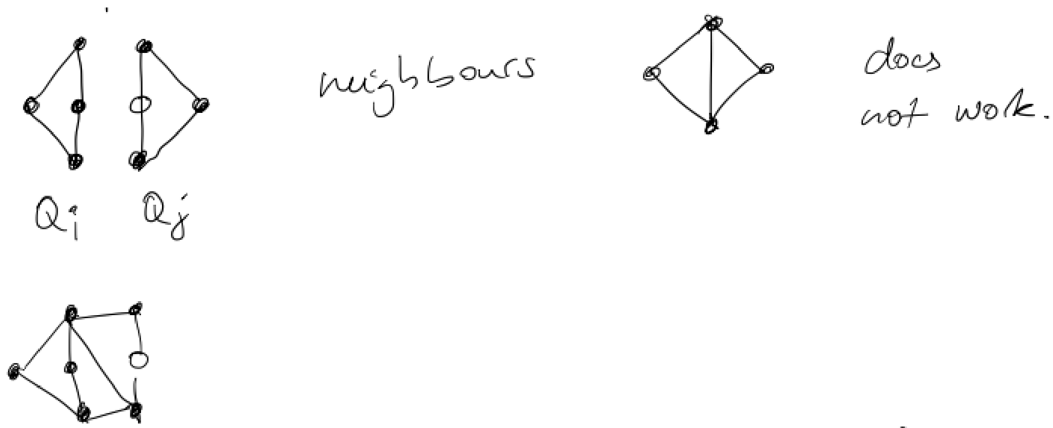


Abbildung 99: subdivision [103]

6.35 Example. See figure 100.

<u>Construction</u>: "roof function": Let $\Psi : A \to \mathbb{R}$,

$$G_\Psi = \mathrm{Conv}\{(w, y) | y \leq \Psi(\omega), \omega \in A, y \in \mathbb{R}\} \subset \mathbb{R}^n \times \mathbb{R},$$

project the upper faces to $\mathbb{R}^d$. The Q_i are defined to be the images of the facets under projection. The upper faces of G_Ψ from the graph of the piecewise affine-linear function

$$g_\Psi : x \mapsto \max\{y : (x, y) \in G_\Psi\}$$

We set $A_i \subset Q_i$ the subset of $A \cap Q_i$ of all points "visible on the roof", i.e. for which $g_\Psi(\omega) = \Psi(\omega)$. (The point $(\omega, \Psi(\omega)) = (\omega, g_\Psi(\omega))$ is then on the boundary of G_Ψ, i.e. visible on the roof, not hidden below.) The marked subdivision constructed like this is denoted by $S(\Psi)$.

[104]Image from Hannah Markwig.

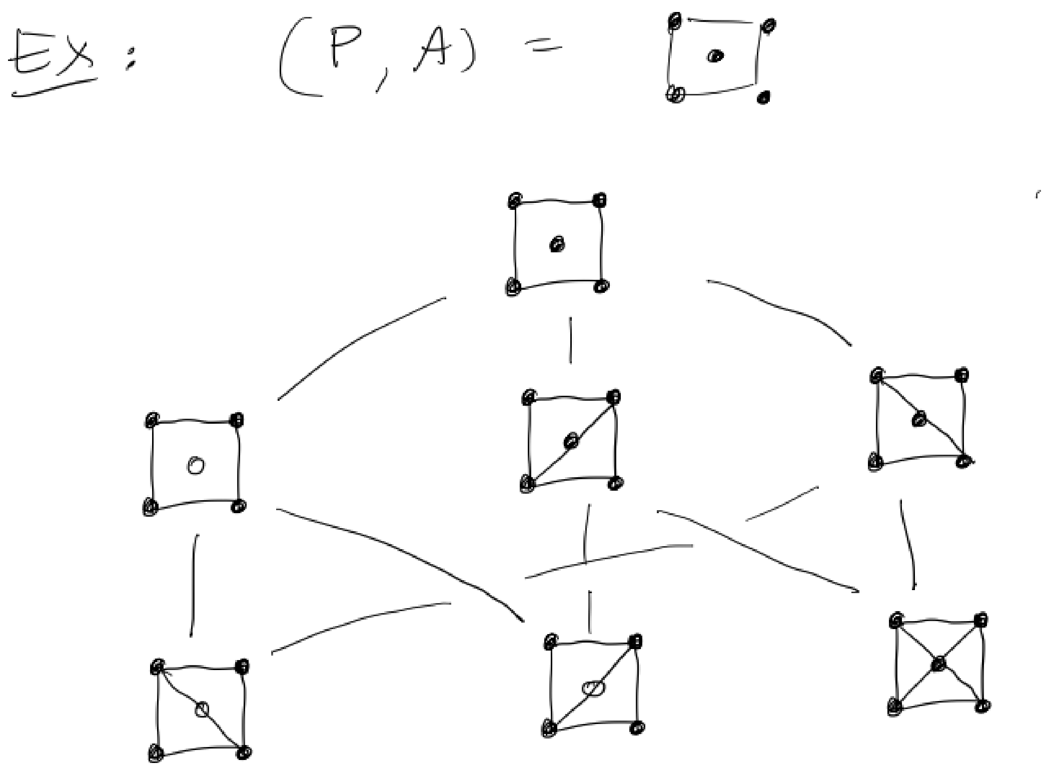


Abbildung 100: refinement [104]

6.36 Example. .

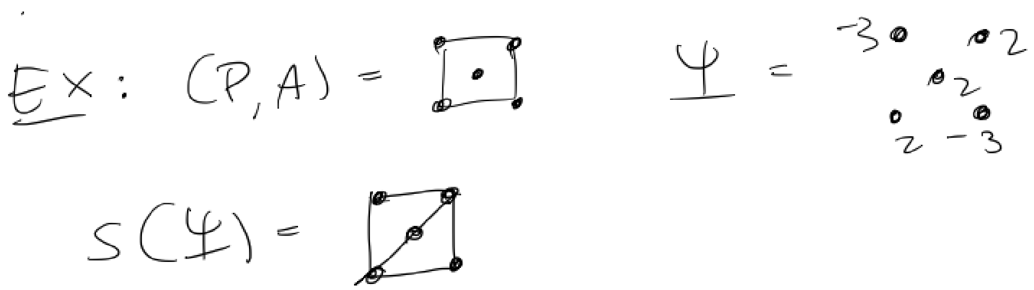


Abbildung 101: Ψ and $S(\Psi)$ [105]

6.37 Definition. A marked subdivision is called **regular**, is it is $S(\Psi)$ for some Ψ. This extends the definition of a regular triangulation.

6.38 Theorem. *(1) The faces of the secondary polytope of* (P, A) *are the convex hulls*

$$F(S) = \mathrm{Conv}(\varphi_T | T \text{ triangulation refining } S)$$

for a regular marked subdivision S.

(2) The cone of the secondary fan corresponding to $F(S)$ *is*

$$C(S) := \{\Psi | S \text{ refines } S(\Psi)\}$$

[105]Image from Hannah Markwig.

(3) $F(S) \subset F(S') \Leftrightarrow S$ refines S'

6.39 Remark. $C(S)$ generalizes our use of $C(T)$ for triangulations T.

Beweis. (1) As before, we consider $(\mathbb{R}^{\#A})^\vee = \{\Psi : A \to \mathbb{R}\}$ and $\mathbb{R}^{\#A} = \{\varphi : A \to \mathbb{R}\}$ to be dual via

$$\langle \Psi, \varphi \rangle = \sum_{\omega \in A} \Psi(\omega) \cdot \varphi(\omega).$$

A function Ψ thus yields a linear functional on the space surrounding the secondary polytope. We consider a supporting affine hyperplane of the secondary polytope of the form

$$H_{\Psi,b} = \{\varphi | \langle \Psi, \varphi \rangle = b\}$$

which cuts out a face Q of the secondary polytope. Q contains all points of the secondary polytope on which Ψ takes its maximum. Thus Q is the convex hull of all vertices φ_T on which Ψ takes its maximum. Let φ_T be such a vertex, then

$$\langle \Psi, \varphi_T \rangle = (n+1) \int_P g_{\Psi,T}(x) dx$$

(because of the lemma about the scalar product, lemma 6.25), and, because it is a maximum

$$\langle \Psi, \varphi_T \rangle = (n+1) \int_P g_{\Psi}(x) dx.$$

Since $g_\Psi(x) \geq g_{\Psi,T}(x) \, \forall \, x \in P$ we conclude $g_\Psi = g_{\Psi,T} \Leftrightarrow T$ refines $S(\Psi)$.

$$\begin{aligned} \Rightarrow Q &= \mathrm{Conv}(\varphi_T | \langle \Psi, \varphi_T \rangle \text{ maximal}) \\ &= \mathrm{Conv}(\varphi_T | T \text{ refines } S(\Psi)) \\ &= F(S(\Psi)) \end{aligned}$$

Vice versa, any $F(S)$ is a face of the secondary polytope.
We can sum up: Ψ takes its maximum on a face $F(S)$ (and not on a bigger face) $\Leftrightarrow F(S) = F(S(\Psi)) \Leftrightarrow S = S(\Psi)$.

(2) Let S be a marked subdivision. Let

$$C(S)^\circ := \{\Psi | S = S(\Psi)\} = \{\Psi | \Psi \text{ takes the maximum at } F(S)\}$$

$\Rightarrow \overline{C(S)^\circ} = \sigma_{F(S)}$, the cone dual to $F(S)$ in the secondary fan. We still need to show that $\overline{C(S)^\circ} = C(S)$. For that, we first show $C(S)$ is closed:
$\Psi \in C(S) \Rightarrow g_{\Psi|Q_i}$ is affine linear $\forall \, Q_i$ in the subdivision S,

$$\begin{aligned} g_\Psi(\omega) &= \Psi(\omega) \text{ must hold } \forall \, \omega \in \bigcup_i A_i, \\ g_\Psi(\omega) &\geq \Psi(\omega) \text{ must hold } \forall \, \omega \notin \bigcup_i A_i. \end{aligned}$$

These are all closed conditions. Furthermore, a general $\Psi \in C(S)$ yields $S(\Psi) = S:$ if it yields a coarser subdivision, one can change the coefficients slightly to get back the finer one, i.e. S. That means that $C(S)$ does not contain more than the closure of $C(S)^\circ$.
$\Rightarrow \sigma_{F(S)} = C(S)$ as claimed.

(3)

$$\begin{aligned}
& F(S) \subset F(S') \\
\Leftrightarrow\, & \sigma_{F(S')} = C(S') = \{\Psi | S' \text{ refines } S(\Psi)\} \subset \sigma_{F(S)} = C(S) = \{\Psi | S \text{ refines } S(\Psi)\} \\
\Leftrightarrow\, & \text{every subdivision refined by } S' \text{ is already refined by } S \\
\Leftrightarrow\, & S \text{ refines } S'.
\end{aligned}$$

□

6.40 Example. Facets of the secondary polytope resp. rays of the secondary fan:

(1) Let $\omega \in A \setminus \{$ vertices of $P\}$, $S = \{(Q, A \setminus \{\omega\})\}$ corresponds to a facet: we obtain a ray of roof functions, since we can vary the height of ω below the roof

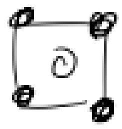


Abbildung 102: ω below the roof [106]

(2) Let g be an affine linear function on $\mathbb{R}^n$, define

$$P_+ = \{x \in P | g(x) \geq 0\}, \quad P_- = \{x | g(x) \leq 0\}.$$

Assume P_+, P_- are full-dimensional. Then $\{(P_\pm, P_\pm \cap A)\}$ defines a facet:
Using lineality vectors of the secondary fan, one can assume that one part of the roof is fixed and horizontal. We can move the position of the other part relative to it.

Abbildung 103: Move the position of the other part [107]

Not all rays are of a form like there two cases.

We study edges of the secondary polytope next.

6.41 Definition. Let (P, A) be a marked polytope, T a triangulation. $Z \subset A$ a circuit s.th. Conv(Z) is fulldimensional $\Leftrightarrow$ $n + 2 = \#Z$.
Assume Conv(Z) is the union of simplices of T having vertices in Z. We know Z has two triangulations T_+ and T_-, so $T|_{\mathrm{Conv}(Z)}$ must coincide with one of those. We define a new triangulation $S_Z(T)$ by replacing this with the other, and keeping all other simplices of T.

[106] Image from Hannah Markwig.
[107] Image from Hannah Markwig.
[108] Image from Hannah Markwig.

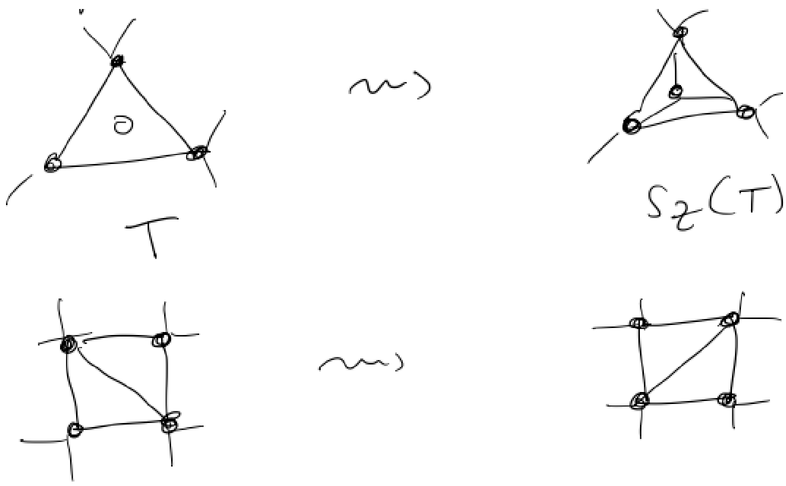


Abbildung 104: $S_Z(T)$ [108]

6.42 Proposition. *With the assumptions from before, assume T and $S_Z(T)$ are regular. Then the vertices φ_T and $\varphi_{S_Z(T)}$ of the secondary polytope are connected by an edge.*

Beweis. Let S be the marked subdivision consisting of $(\mathrm{Conv(Z)}, \mathrm{Z})$ and the marked simplices that T and $S_Z(T)$ share. Since T and $S_Z(T)$ are regular, also S is (it is given by a "flattening " of the roof function of T resp. $S_Z(T)$).
More precisely, if we want all the points of Z to be contained in a piece of an affine hyperplane in the graph of the roof function, we obtain one linear condition on the heights of $\Psi \Rightarrow C(S)$ is of codimension $1 \Rightarrow F(S)$ is an edge of the secondary polytope connecting the two vertices. □

The general case of edges is similar, but more notation is required:

6.43 Definition. Let T be a triangulation of (P, A), $Z \subset A$ circuit. We call Z a **part** of T, if

(1) T has no vertices in $\mathrm{Conv}(Z)$ except Z

(2) $\mathrm{Conv}(Z)$ is a union of simplices in T

(3) If $\mathrm{Conv}(I)$, $\mathrm{Conv}(I')$ are two maximal simplices of one of the two triangulations of $(\mathrm{Conv}(Z), Z)$, and $F \subset A \setminus Z$, then

$$\mathrm{Conv}(I \cup F) \text{ in } T \Leftrightarrow \mathrm{Conv}(I' \cup F) \text{ in } T.$$

6.44 Remark. If $\mathrm{Conv}(Z)$ is fulldimensional, 3) is no further condition, as it requires $F \neq 0$.

6.45 Example. We look at Z and (P, A) in figure 105.

To see this, we check 3): for F, we can take any of the upper or lower vertex. In the left picture, there is only one simplex $\mathrm{Conv}(I)$. In the right picture, both triangles on the lower half belong to T (for F the lower vertex).

[109]Image from Hannah Markwig.

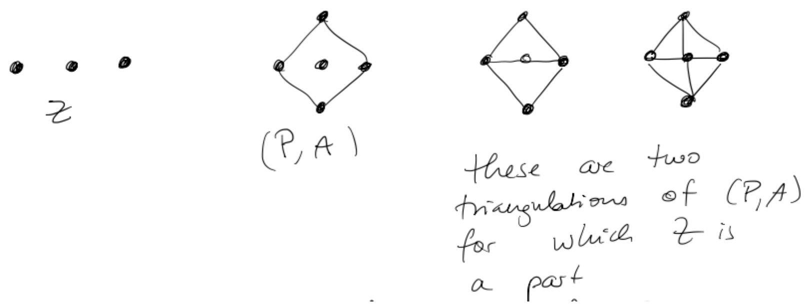


Abbildung 105: Z, (P, A) and two triangulations [109]

6.46 Definition. Let Z be a part of T, then T induces one of the two triangulations of $\mathrm{Conv}(Z)$ on it, say T_+.
We let $S_Z(T)$ be the triangulation which substitutes all simplices of the form $\mathrm{Conv}(I \cup F)$ with $\mathrm{Conv}(I) \in T_+$ by all of the form $\mathrm{Conv}(I' \cup F)$ with $\mathrm{Conv}(I') \in T_-$.
We say, $S_Z(T)$ is obtained from T by **flipping** Z.

6.47 Remark. $S_Z(S_Z(T)) = T$

6.48 Theorem. *Let T, T' be triangulations of (P, A). The vertices φ_T and $\varphi_{T'}$ of the secondary polytope are connected by an edge $\Leftrightarrow \exists$ circuit Z which is a part of T and T' s.th. $T = S_Z(T')$.*

The idea for the proof is similar to before: we need to describe the subdivision S corresponding to the edge.

6.49 Definition. $J \subset A \setminus Z$ **separates** T and T' $:\Leftrightarrow \exists\, \omega \in Z$ s.th. $Z \setminus \{\omega\} \cup J$ is the set of vertices of a simplex of T resp. T' of maximal dimension.
The subdivision of the edge $\overline{\varphi_T \varphi_{T'}}$ consists of simplices $(\mathrm{Conv}(I), I)$ that appear both in T and T', and simplices of the form $(\mathrm{Conv}(Z \cup J), Z \cup J)$ for a separating set $J \subset A \setminus Z$.
If $\mathrm{Conv}(Z)$ is fulldimensional, $\emptyset$ is the only separating set.

6.50 Example. .

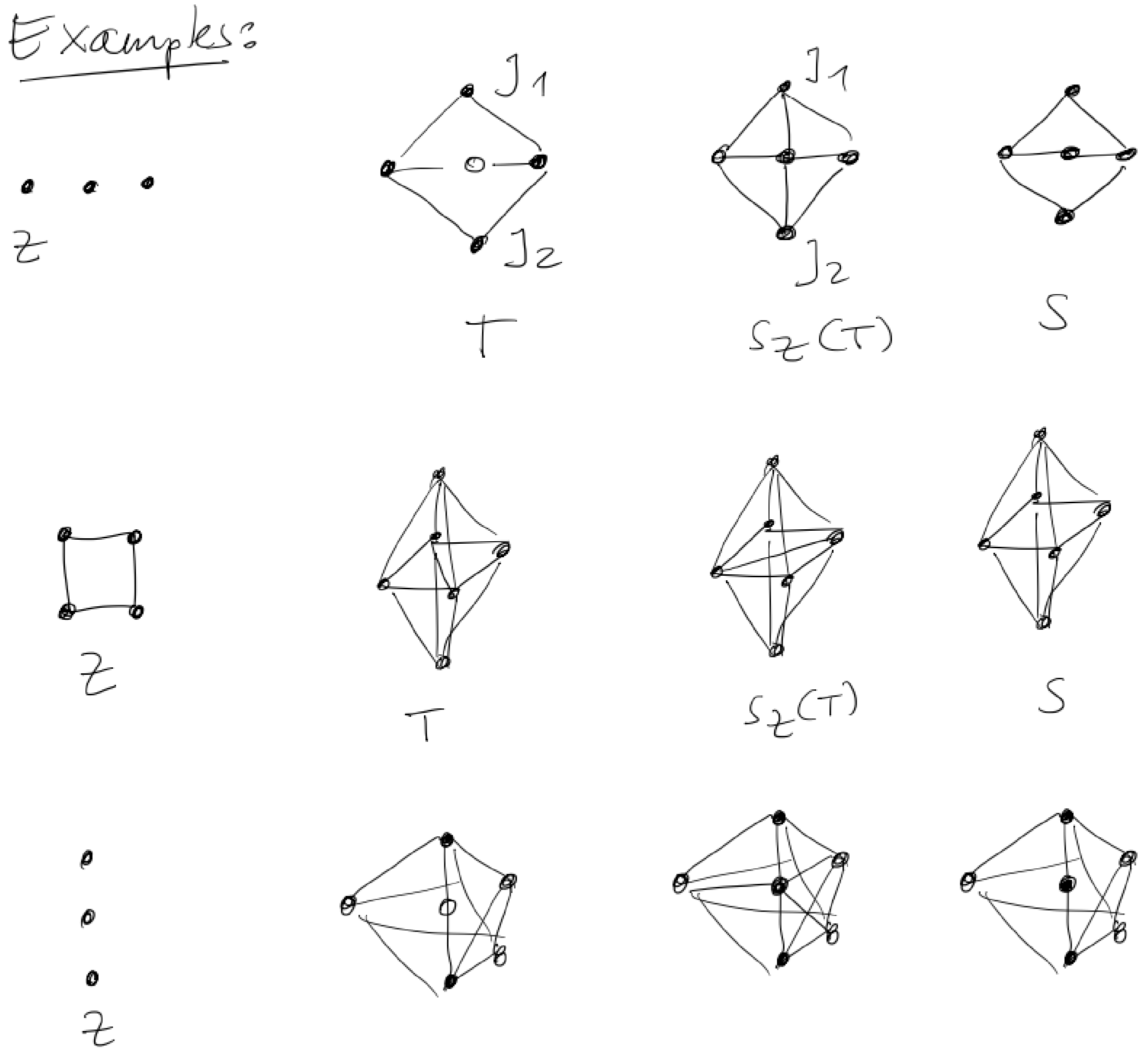


Abbildung 106: Z, T, $S_Z(T)$ and S [110]

Question: What are choices for weights that make the codimension−1−skeleton of the secondary fan balanced?

[110]Image from Hannah Markwig.

7 Ehrhart Theory

Ehrhart theory deals with counts of lattice points inside stretched lattice polytopes. We start with a case study: the d-dimensional cube.

7.1 Definition. Let P be the d-dimensional cube,

$$P = [0,1]^d = \{(x_1, \dots, x_d) \in \mathbb{R}^d | 0 \leq x_i \leq 1 \,\forall\, i\}.$$

The cube has $2d$ facets given by the hyperplanes $x_1 = 0, x_1 = 1, \dots, x_d = 0, x_d = 1$.
We stretch the cube (respectively, any polytope P) with a factor $t \in \mathbb{N}_{\geq 0}$:

$$tP = \{(tx_1, \dots, tx_d) | (x_1, \dots, x_d) \in P\}$$

We count:

$$\#(tP \cap \mathbb{Z}^d) = \#[0,t]^d \cap \mathbb{Z}^d = (\#[0,t] \cap \mathbb{Z})^d = (t+1)^d.$$

7.2 Definition. For a lattice polytope $P \subset \mathbb{R}^d$, we consider the **lattice point count** of the t-th stretch as the function

$$L_P(t) := \#(tP \cap \mathbb{Z}^d).$$

We can also imagine that we leave P fixed and make the lattice finer and finer:

$$L_P(t) = \#(P \cap \frac{1}{t}\mathbb{Z}^d)$$

In this sense, we can view $L_P(t)$ as the **discrete volume** of P.
Back to the cube:

$$L_P(t) = (1+t)^d$$

is a polynomial in t, its coefficients are the binomial coefficients

$$(1+t)^d = \sum_{k=0}^{d} \binom{d}{k} t^k.$$

We now consider lattice points in the interior of the cube:

$$L_{P^\circ}(t) = \#(tP^\circ \cap \mathbb{Z}^d) = \#((0,t)^d \cap \mathbb{Z}^d) = (\#(0,t) \cap \mathbb{Z})^d = (t-1)^d.$$

<u>Note:</u> For the cube P we have $(-1)^d \cdot L_P(-t) = L_{P^\circ}(t)$

7.3 Definition. For a lattice polytope $P \subset \mathbb{R}^d$, we define the **Ehrhart series** of P as the generating series of the lattice point count:

$$\mathrm{Ehr}_P(z) = 1 + \sum_{t=1}^{\infty} L_P(t) z^t$$

To understand the Ehrhart series of a cube, we define **Euler numbers:**

7.4 Definition. We define $A(d,k)$ via

$$\sum_{j\geq 0} j^d z^j = \frac{\sum_{k=0}^d A(d,k) z^k}{(1-z)^{d+1}}.$$

The geometric series

$$\frac{1}{1-z} = \sum_{j\geq 0} z^j$$

can be derived and multiplied by z:

$$z \cdot \frac{d}{dz}\Big(\frac{1}{1-z}\Big) = \sum_{j\geq 0} j z^j$$

If we do it d times, we obtain

$$\Big(z \cdot \frac{d}{dz}\Big)^d \Big(\frac{1}{1-z}\Big) = \sum_{j\geq 0} j^d z^j = \frac{\sum_{k=0}^d A(d,k) z^k}{(1-z)^{d+1}}$$

7.5 Example. <u>d=0:</u>

$$\sum_{j\geq 0} j^0 z^j = \sum_{j\geq 0} z^j = \frac{1}{(1-z)^{0+1}} \qquad \Rightarrow A(0,0) = 1$$

<u>d=1:</u>

$$\begin{aligned}\sum_{j\geq 0} j z^j &= z \cdot \frac{d}{dz}\Big(\frac{1}{1-z}\Big) \overset{\text{quotient rule}}{=} z \cdot \frac{1}{(1-z)^2} = \frac{A(1,0) + A(1,1)z}{(1-z)^2} \\ &\Rightarrow A(1,0) = 0,\ A(1,1) = 1\end{aligned}$$

<u>d=2:</u>

$$\begin{aligned}\sum_{j\geq 0} j^2 z^j &= \Big(z \cdot \frac{d}{dz}\Big)^2 \Big(\frac{1}{1-z}\Big) = z \cdot \frac{d}{dz}\Big(\frac{z}{(1-z)^2}\Big) = z \cdot \frac{(1-z)^2 - z \cdot 2(1-z)(-1)}{(1-z)^4} \\ &= z \cdot \frac{(1-z) + 2z}{(1-z)^3} = \frac{z + z^2}{(1-z)^3} \\ &\Rightarrow A(2,0) = 0,\ A(2,1) = 1,\ A(2,2) = 1\end{aligned}$$

<u>d=3:</u>

$$\begin{aligned}\sum_{j\geq 0} j^3 z^j &= \Big(z \cdot \frac{d}{dz}\Big)\Big(\frac{z + z^2}{(1-z)^3}\Big) = z \cdot \frac{(1+2z)(1-z)^3 - (z+z^2) \cdot 3(1-z)^2(-1)}{(1-z)^6} \\ &= z \cdot \frac{(1+2z)(1-z) + 3(z+z^2)}{(1-z)^4} = z \cdot \frac{1 + z - 2z^2 + 3z + 3z^2}{(1-z)^4} = \frac{z + 4z^2 + z^3}{(1-z)^4} \\ &\Rightarrow A(3,0) = 0,\ A(3,1) = 1,\ A(3,2) = 4,\ A(3,3) = 1\end{aligned}$$

7.6 Lemma. *The Euler numbers $A(d,k)$ satisfy:*

(1) $A(d,k) = (d-k+1) \cdot A(d-1,k-1) + k \cdot A(d-1,k)$

(2) $A(d,k) = A(d, d+1-k)$

(3) $A(d,k) = \sum_{j=0}^{k} (-1)^j \binom{d+1}{j} (k-j)^d$

(4) $\sum_{k=0}^{d} A(d,k) = d!$

Beweis. (1)

$$
\begin{aligned}
&z \cdot \frac{d}{dz}\left(\frac{\sum_{k=0}^{d-1} A(d-1,k) z^k}{(1-z)^{d-1+1}}\right) \\
=& z \cdot \Big(\sum_{k=1}^{d-1} A(d-1,k) \cdot k \cdot z^{k-1}(1-z)^d - \sum_{k=0}^{d-1} A(d-1,k) z^k d (1-z)^{d-1} \cdot (-1)\Big) \\
&\cdot \frac{1}{(1-z)^{2d}} \\
=& \Big(\sum_{k=1}^{d-1} A(d-1,k) \cdot k \cdot z^k (1-z) + \sum_{k=0}^{d-1} A(d-1,k) z^{k+1} \cdot d\Big) \cdot \frac{1}{(1-z)^{d+1}} \\
=& \Big(\sum_{k=1}^{d-1} A(d-1,k) \cdot k \cdot z^k - \sum_{k=1}^{d-1} A(d-1,k) \cdot k \cdot z^{k+1} + \sum_{k=0}^{d-1} A(d-1,k) \cdot d \cdot z^{k+1}\Big) \\
&\cdot \frac{1}{(1-z)^{d+1}} \\
\overset{k'=k+1}{=}& \Big(\sum_{k=0}^{d-1} A(d-1,k) \cdot k \cdot z^k - \sum_{k'=1}^{d} A(d-1,k'-1) \cdot (k'-1) \cdot z^{k'} \\
&+ \sum_{k'=1}^{d} A(d-1,k'-1) \cdot d \cdot z^{k'}\Big) \cdot \frac{1}{(1-z)^{d+1}} \\
\underset{A(d-1,0)=0}{\overset{\text{rename } k'=k}{=}}& \Big(\sum_{k=0}^{d} (k \cdot A(d-1,k) + (d-k+1) \cdot A(d-1,k-1)) \cdot z^k\Big) \cdot \frac{1}{(1-z)^{d+1}} \\
\Rightarrow& A(d,k) = k \cdot A(d-1,k) + (d-k+1) \cdot A(d-1,k-1)
\end{aligned}
$$

(2) We use induction on d. Note that for $d = 1$, $A(1,0) = 0$, $A(1,1) = 1$. We can use $d = 1, k = 1$ and obtain $1 = A(1,1) = A(1, 1+1-1)$.

$$
\begin{aligned}
A(d, d+1-k) &\overset{1)}{=} (d - (d+1-k) + 1) A(d-1, d-k) + (d-k+1) A(d-1, d-k+1) \\
&= k \cdot A(d-1, d-k) + (d-k+1) \cdot A(d-1, d-k+1) \\
&\overset{\text{induction}}{=} k \cdot A(d-1, d-(d-k)) + (d-k+1) \cdot A(d-1, d-(d-k+1)) \\
&= k \cdot A(d-1, k) + (d-k+1) \cdot A(d-1, k-1) \\
&\overset{1)}{=} A(d,k)
\end{aligned}
$$

(3) Again, we use induction on d. For $d = 1$, $A(1,0) = 0$, $A(1,1) = 1$.

$$A(1,0) = 0 = \text{ empty sum } \checkmark$$

$$A(1,1) = \sum_{j=0}^{1} (-1)^j \binom{1}{j} (1-j)^2 = 1 + 0 = 1 \checkmark$$

$$
\begin{aligned}
A(d,k) &\overset{1)}{=} k\cdot A(d-1,k) + (d-k+1)\cdot A(d-1,k-1) \\
&\overset{\text{induction}}{=} k\cdot \sum_{j=0}^{k}(-1)^j\binom{d}{j}(k-j)^{d-1} + (d-k+1)\cdot \sum_{j=0}^{k-1}(-1)^j\binom{d}{j}(k-1-j)^{d-1} \\
&\underset{j=j'-1}{\overset{j'=j+1}{=}} k\cdot \sum_{j=0}^{k}(-1)^j\binom{d}{j}(k-j)^{d-1} + (d-k+1)\cdot \sum_{j'=1}^{k}(-1)^{j'-1}\binom{d}{j'-1}(k-j')^{d-1} \\
&\underset{j=j'}{\overset{\text{rename}}{=}} k\cdot \sum_{j=0}^{k}(-1)^j\binom{d}{j}(k-j)^{d-1} + (d-k+1)\cdot \sum_{j=1}^{k}(-1)^{j-1}\binom{d}{j-1}(k-j)^{d-1} \\
&= \sum_{j=0}^{k}(-1)^j\left(k\cdot\binom{d}{j} - (d-k+1)\binom{d}{j-1}\right)(k-j)^{d-1} \\
&= \sum_{j=0}^{k}(-1)^j\left(\frac{k\cdot d!}{(d-j)!j!} - \frac{(d-k+1)d!}{(d-j+1)!(j-1)!}\right)(k-j)^{d-1} \\
&= \sum_{j=0}^{k}(-1)^j\left(\frac{d!(k\cdot(d-j+1)-(d-k+1)j}{(d-j+1)!j!}\right)(k-j)^{d-1} \\
&= \sum_{j=0}^{k}(-1)^j\left(\frac{d!kd-kj+k-(dj-kj+j)}{(d-j+1)!j!}\right)(k-j)^{d-1} \\
&= \sum_{j=0}^{k}(-1)^j\left(\frac{(d+1)!}{(d-j+1)!j!}\right)(k-j)^{d} \\
&= \sum_{j=0}^{k}(-1)^j\binom{d+1}{j}(k-j)^{d}
\end{aligned}
$$

(4) We use induction on d.

$$
\begin{aligned}
&A(1,0)+A(1,1) = 0+1 = 1 = 1! \\
&A(2,0)+A(2,1)+A(2,2) = 0+1+1 = 2 = 2! \\
&A(3,0)+A(3,1)+A(3,2)+A(3,3) = 0+1+4+1 = 6 = 3!
\end{aligned}
$$

We would like to insert $z=1$ into the numerator of our generating function to obtain $\sum_{k=0}^{d} A(d,k)$.
We cannot insert $z=1$ into our whole generating function, since it does not converge.
We get around this by using induction:

$$
\begin{aligned}
&z\cdot\frac{d}{dz}\left(\frac{\sum_{k=0}^{d-1}A(d-1,k)z^k}{(1-z)^{d-1+1}}\right) \\
&\overset{\text{see 1)}}{=}\left(\sum_{k=1}^{d-1}A(d-1,k)\cdot k\cdot z^k(1-z) + \sum_{k=0}^{d-1}A(d-1,k)z^{k+1}\cdot d\right)\cdot\frac{1}{(1-z)^{d+1}}
\end{aligned}
$$

The numerator is

$$
\sum_{k=1}^{d-1}A(d-1,k)\cdot k\cdot z^k(1-z) + \sum_{k=0}^{d-1}A(d-1,k)z^{k+1}\cdot d
$$

Now we insert $z=1$ and obtain

$$
\sum_{k=0}^{d-1}A(d-1,k)\cdot d \overset{\text{induction}}{=} (d-1)!\cdot d = d!
$$

$\square$

7.7 Remark. Using statement 1) of lemma 7.6, we can arrange the $A(d,k)$ into a triangular shape similar to Pascal's triangle:

$$
\begin{tikzcd}
A(d-1,k-1)\cdot(d-k+1) \arrow[dr] & {} \arrow[d, phantom, "+"] & A(d-1,k)\cdot k \arrow[dl] \\
& A(d,k) &
\end{tikzcd}
$$

$$
\begin{tikzcd}
& k=0 & k=1 & k=2 & k=3 & & & \\
d=0 & 1 & & & & & & \\
d=1 & 0 & 1 & & & & & \\
d=2 & 0 & 1 \arrow[dr, "\times 2"] & 1 \arrow[d, "\times 2"'] & & & & \\
d=3 & 0 & 1 \arrow[dr, "\times 3"] & 4 \arrow[d, "\times 2"'] \arrow[dr, "\times 2"] & 1 \arrow[d, "\times 3"'] & & & \\
d=4 & 0 & 1 & 11 & 11 & 1 & & \\
d=5 & 0 & 1 & 26 & 66 & 26 & 1 & \\
d=6 & 0 & 1 & 57 & 302 & 302 & 57 & 1
\end{tikzcd}
$$

We can now express the Ehrhart series of the cube P via Euler numbers:

$$
\begin{aligned}
\mathrm{Ehr}_P(z) =& 1 + \sum_{t\geq 1}(t+1)^d z^t = \sum_{t\geq 0}(t+1)^d z^t \\
\overset{t'=t+1}{=}& \sum_{t'\geq 1} t'^d z^{t'-1} \underset{t'=t}{\overset{\text{rename}}{=}} \frac{1}{z}\cdot\sum_{t\geq 1} t^d z^t \\
=& \frac{\sum_{k=1}^d A(d,k) z^{k-1}}{(1-z)^{d+1}}
\end{aligned}
$$

We sum up our results for the cube:

7.8 Theorem. *Let P be the d-dim cube.*

(1) The lattice point count is the polynomial

$$L_P(t) = (t+1)^d = \sum_{k=0}^{d}\binom{d}{k} t^k$$

(2) Evaluated at $-t$, we obtain the count of interior lattice points up to sign: $(-1)^d L_P(-t) = L_{P^\circ}(t)$

(3) The Ehrhart series is

$$\mathrm{Ehr}(z) = \frac{\sum_{k=1}^{d} A(d,k) z^{k-1}}{(1-z)^{d+1}}$$

The next case study is the standard simplex

$$\triangle = \mathrm{Conv}(0, e_1, \ldots, e_d) = \{(x_1, \ldots, x_d) \in \mathbb{R}^d : \sum x_i \leq 1, x_k \geq 0\}$$
$$t\triangle = \{(x_1, \ldots, x_d) : \sum x_i \leq t, x_k \geq 0\}$$

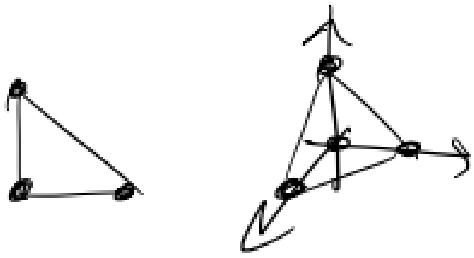


Abbildung 107: Standard simplex $\triangle$ [111]

To avoid having to deal with the first inequality, we introduce another variable:
We want to count integer points $(m_1, \ldots, m_d)$ satisfying $m_i \geq 0$ and $m_1 + \ldots + m_d \leq t$. Instead, we count solutions $(m_1, \ldots, m_d, m_{d+1}) \in \mathbb{Z}_{\geq 0}^{d+1}$ satisfying $m_1 + \ldots + m_d + m_{d+1} = t$. Then

$$\begin{aligned} L_\triangle(t) =& \#(t\triangle \cap \mathbb{Z}^d) = \#\{(m_1, \ldots, m_{d+1}) \in \mathbb{Z}_{\geq 0}^{d+1} | m_1 + \ldots + m_{d+1} - t = 0\} \\ =& \mathrm{Const}\Big(\Big(\sum_{m_1 \geq 0} z^{m_1}\Big) \cdot \ldots \cdot \Big(\sum_{m_{d+1} \geq 0} \cdot z^{-t}\Big)\Big) \\ =& \mathrm{Const}\Big(\frac{1}{(1-z)^{d+1} \cdot z^t}\Big) \end{aligned}$$

<u>Claim:</u>

$$\frac{1}{(1-z)^{d+1}} = \sum_{k \geq 0} \binom{d+k}{d} z^k$$

(binomial series)

$$\frac{1}{(1-z)^{d+1}} = \frac{1}{1-z} \cdot \ldots \cdot \frac{1}{1-z} = \sum_{j_1 \geq 0} z^{j_1} \cdot \ldots \cdot \sum_{j_{d+1} \geq 0} z^{j_{d+1}}$$

To obtain the coefficient of z^k of this product, we have to take all choices of $j_1, \ldots, j_{d+1}$ s.th. their sum is k.$\rightsquigarrow$ Combinatorically, we can view this as the count of possible ways to separate $d + k$ places with d walls (figure 108)

[111] Image from Hannah Markwig.

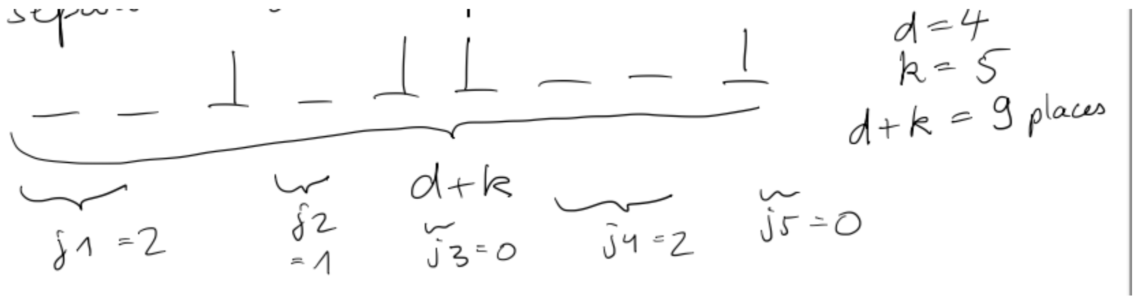


Abbildung 108: $d+k$ places, d walls [112]

$$
\begin{aligned}
&j_1+j_2+j_3+j_4+j_5 = 2+1+0+2+0 = 5\\
=\,&\#\text{ places not filled with a wall}\\
=\,&d+k-\#\text{ walls } = d+k-d = k
\end{aligned}
$$

There are $\binom{d+k}{d}$ choices to put walls on the $d+k$ places

$$\Rightarrow \operatorname{Coeff}_{z^k}\left(\frac{1}{(1-z)^{d+1}}\right) = \binom{d+k}{k}.$$

For the lattice point count of the simplex $\triangle$:

$$L_\triangle(t) = \#(t\triangle \cap \mathbb{Z}^d)$$

$$
\begin{aligned}
\operatorname{Const}\left(\frac{1}{(1-z)^{d+1}}\cdot\frac{1}{z^t}\right) &= \operatorname{Const}\left(\sum_{k\geq 0}\binom{d+k}{k}z^{k-t}\right)\\
&= \operatorname{Coeff}_{z^t}\left(\sum_{k\geq 0}\binom{d+k}{k}z^{k}\right) = \binom{d+t}{d}\\
&= \frac{1}{d!}\underbrace{(d+t)\cdot(d+t-1)\ldots(t+1)}_{d\text{ factors}}
\end{aligned}
$$

is a polynomial of degree d in t.

[112] Image from Hannah Markwig.

Next, we count interior lattice points of the simplex:

$$\begin{aligned}
L_{\triangle^\circ}(t) &= \#\{(m_1,\ldots,m_d)\in\mathbb{Z}^d_{>0} : m_1+\ldots+m_d<t\}\\
&= \#\{(m_1,\ldots,m_{d+1})\in\mathbb{Z}^{d+1}_{>0} : m_1+\ldots+m_{d+1}=t\}\\
&= \operatorname{Const}\Big(\Big(\sum_{m_1>0} z^{m_1}\Big)\cdot\ldots\cdot\Big(\sum_{m_{d+1}>0} z^{m_{d+1}}\Big)\cdot z^{-t}\Big)\\
&= \operatorname{Const}\Big(\Big(\frac{z}{1-z}\Big)^{d+1} z^{-t}\Big)\\
&= \operatorname{Const}\Big(z^{d+1-t}\cdot\frac{1}{(1-z)^{d+1}}\Big)\\
&= \operatorname{Coeff}_{t-1-d}\Big(\frac{1}{(1-z)^{d+1}}\Big)\\
&= \operatorname{Coeff}_{t-1-d}\Big(\sum_{k\geq 0}\binom{d+k}{d} z^k\Big)\\
&= \binom{d+t-1-d}{d} = \binom{t-1}{d}
\end{aligned}$$

We sum up our results:

7.9 Theorem. *Let $\triangle\subset\mathbb{R}^d$ be the standard simplex. Then:*

(1) The lattice point count equals

$$L_\triangle(t) = \binom{d+t}{d} = \frac{1}{d!}\underbrace{(d+t)\cdot(d+t-1)\cdot\ldots\cdot(t+1)}_{d\text{ factors}}$$

(2) $(-1)^d\cdot L_\triangle(-t) = L_{\triangle^\circ}(t)$

(3) $\operatorname{Ehr}_\triangle(z) = \frac{1}{(1-z)^{d+1}}$

Beweis. (1) see before.

(2)

$$\begin{aligned}
(-1)^d L_\triangle(-t) &= (-1)^d\binom{d-t}{d}\\
&= (-1)^d\frac{1}{d!}(d-t)\cdot(d-t-1)\cdot\ldots\cdot(-t+1)\\
&= \frac{1}{d!}(t-d)\cdot(t-d+1)\cdot\ldots\cdot(t-1)\\
&= \frac{1}{d!}(t-1)\cdot\ldots\cdot(t-d)\\
&= \frac{(t-1)!}{(t-d-1)!d!}\\
&= \binom{t-1}{d} = L_{\triangle^\circ}(t)
\end{aligned}$$

by our previous computation.

(3)

$$\mathrm{Ehr}_{\triangle}(z) = 1 + \sum_{t\geq 1} L_{\triangle}(t) z^t = 1 + \sum_{t\geq 1} \binom{d+t}{d} z^t = \sum_{t\geq 0} \binom{d+t}{d} z^t = \frac{1}{(1-z)^{d+1}}$$

□

Our next case study:
Pyramids over the $(d-1)$-dim cube with vertex at e_d:

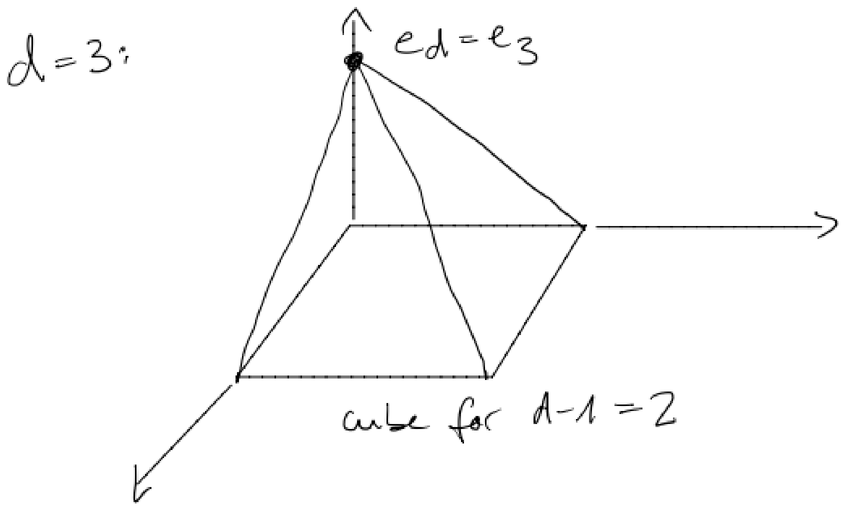


Abbildung 109: Pyramid [113]

For those, we need **Bernoulli polynomials**. They are given via the generating function

$$\frac{ze^{xz}}{e^z - 1} = \sum_{k\geq 0} \frac{\mathrm{B}_k(x)}{k!} z^k$$

Remember the exponential series:

$$e^y = \sum_{k\geq 0} \frac{y^k}{k!}$$

[113]Image from Hannah Markwig.

We compute the first two Bernoulli polynomials:

$$\begin{aligned}
\frac{ze^{xz}}{e^z-1} &= \frac{z}{e^z-1}\cdot e^{xz} \\
&= \frac{z}{z+\frac{z^2}{2!}+\frac{z^3}{3!}+\dots}\cdot e^{xz} \\
&= \frac{z+\frac{z^2}{2!}+\frac{z^3}{3!}+\dots-\frac{z^2}{2!}-\frac{z^3}{3!}-\dots}{z+\frac{z^2}{2!}+\frac{z^3}{3!}+\dots}\cdot\sum_{k\geq 0}\frac{(xz)^k}{k!} \\
&= 1-\frac{\frac{z^2}{2!}+\frac{z^3}{3!}+\dots}{z+\frac{z^2}{2!}+\frac{z^3}{3!}+\dots}\cdot\sum_{k\geq 0}\frac{(xz)^k}{k!} \\
&= (1-\frac{z}{2}+\dots)\cdot\sum_{k\geq 0}\frac{(xz)^k}{k!} \\
&= \sum_{k\geq 0}\frac{(xz)^k}{k!}-\frac{1}{2}\sum_{k\geq 0}\frac{x^k z^{k+1}}{k!}+\dots \\
&\Rightarrow \mathrm{B}_0(x)=1,\ \mathrm{B}_1(x)=x-\frac{1}{2}
\end{aligned}$$

One can further show that

$$\begin{aligned}
\mathrm{B}_2(x) &= x^2-x+\frac{1}{6}, \\
\mathrm{B}_3(x) &= x^3-\frac{3}{2}x^2+\frac{1}{2}x,\ \dots
\end{aligned}$$

7.10 Definition. The **Bernoulli numbers** are $\mathrm{B}_k := \mathrm{B}_k(0)$ and have the generating series

$$\frac{z}{e^z-1}=\sum_{k\geq 0}\frac{\mathrm{B}_k}{k!}z^k$$

7.11 Lemma. *Let $d \geq 1$, $n \geq 2$ be integers.*

$$\frac{1}{d}\cdot(\mathrm{B}_d(n)-\mathrm{B}_d)=\sum_{k=0}^{n-1}k^{d-1}$$

Beweis. We consider the generating series of $\frac{\mathrm{B}_d(n)-\mathrm{B}_d}{d!}$:

$$\begin{aligned}
\sum_{d\geq 0}\frac{\mathrm{B}_d(n)-\mathrm{B}_d}{d!}z^d &= \sum_{d\geq 0}\frac{\mathrm{B}_d(n)}{d!}z^d=\sum_{d\geq 0}\frac{\mathrm{B}_d(n)}{d!}z^d-\sum_{d\geq 0}\frac{\mathrm{B}_d}{d!}z^d \\
&= \frac{ze^{nz}}{e^z-1}-\frac{z}{e^z-1}=\frac{z(e^{nz}-1)}{e^z-1} \\
&= z\cdot\frac{(e^z)^n-1}{e^z-1}=z\cdot\sum_{k=0}^{n-1}e^{kz} \\
&= z\cdot\sum_{k=0}^{n-1}\sum_{j\geq 0}\frac{(kz)^j}{j!}=\sum_{j\geq 0}\left(\sum_{k=0}^{n-1}k^j\right)\frac{z^{j+1}}{j!} \\
&\overset{j'=j+1}{=} \sum_{j'\geq 1}\left(\sum_{k=0}^{n-1}k^{j'-1}\right)\frac{z^{j'}}{(j'-1)!} \\
&\underset{j'=j}{\overset{\text{rename}}{=}} \sum_{j\geq 1}\left(\sum_{k=0}^{n-1}k^{j-1}\right)\frac{z^j}{(j-1)!}.
\end{aligned}$$

When comparing coefficients, we obtain the result. $\square$

Now back to our pyramid:

$$P = \{(x_1, \ldots, x_d) \in \mathbb{R}^d | 0 \leq x_1, \ldots, x_{d-1} \leq 1 - x_d \leq 1\}$$

$$L_P(t) = \#\{(m_1, \ldots, m_d) \in \mathbb{Z}^d | 0 \leq m_k \leq t - m_d \leq t \,\forall\, k = 1, \ldots, d-1\}$$

If we fix m_d between 0 and t, we can choose $t - m_d + 1$ numbers for each of the m_k independently. Thus

$$L_P(t) = \sum_{m_d=0}^{t} (t - m_d + 1)^{d-1} \overset{k=t-m_d+1}{=} \sum_{k=1}^{t+1} k^{d-1} = \frac{1}{d}(\mathrm{B}_d(t+2) - \mathrm{B}_d) \quad \text{(polynomial in } t\text{)}$$

We count interior lattice points for the pyramid:

$$L_{P^\circ}(t) = \#\{(m_1, \ldots, m_d) \in \mathbb{Z}^d | 0 < m_k < t - m_d < t \text{ for } k = 1, \ldots, d-1\}$$

Similarly,

$$L_{P^\circ}(t) = \sum_{m_d=1}^{t-1} (t - m_d - 1)^{d-1} = \sum_{k=0}^{t-2} k^{d-1} = \frac{1}{d}(\mathrm{B}_d(t-1) - \mathrm{B}_d).$$

7.12 Lemma.

$$\mathrm{B}_d(1-x) = (-1)^d \mathrm{B}_d(x)$$

Beweis.

$$\begin{aligned}\sum_{k\geq 0} \frac{\mathrm{B}_k(1-x)}{k!} z^k &= \frac{ze^{(1-x)z}}{e^z - 1} = \frac{ze^z e^{-xz}}{e^z(1-e^{-z})} \\ &= \frac{ze^{-xz}}{(1-e^{-z})} = \frac{(-z) \cdot e^{x(-z)}}{e^{-z} - 1} \\ &= \sum_{k\geq 0} \frac{\mathrm{B}_k(x)}{k!} (-z)^k,\end{aligned}$$

the result follows by comparing coefficients. $\square$

7.13 Lemma. $\mathrm{B}_d = 0$ *for all odd* $d \geq 3$.

Beweis. $\mathrm{B}_0 = 1$, $\mathrm{B}_1 = -\frac{1}{2}$.

$$\sum_{k\geq 0} \frac{\mathrm{B}_k}{k!} z^k = 1 - \frac{1}{2} z + \sum_{k\geq 2} \frac{\mathrm{B}_k}{k!} z^k = \frac{z}{e^z - 1}$$

$$\Rightarrow \sum_{\substack{k\geq 2 \\ \text{and } k=0}} \frac{\mathrm{B}_k}{k!} z^k = \frac{z}{e^z - 1} + \frac{1}{2} z = \frac{(z + \frac{1}{2} z e^z - \frac{1}{2} z)}{e^z - 1} = \frac{z(1 + \frac{1}{2} e^z - \frac{1}{2})}{e^z - 1} = \frac{z \cdot \frac{1}{2} \cdot (e^z + 1)}{e^z - 1}$$

We claim that this is an even function. We insert $(-z)$:

$$\frac{(-z) \cdot \frac{1}{2} \cdot (e^{-z} + 1)}{e^{(-z)} - 1} = \frac{e^z(-z) \cdot \frac{1}{2} \cdot (e^{-z} + 1)}{e^z \cdot (e^{(-z)} - 1)} = \frac{(-z) \cdot \frac{1}{2} \cdot (1 + e^z)}{1 - e^z} = \frac{z \cdot \frac{1}{2}(1 + e^z)}{e^z - 1}$$

$\square$

7.14 Proposition. *For the pyramid P:*

$$(-1)^d L_P(-t) = L_{P^\circ}(t)$$

Beweis.

$$\begin{aligned}(-1)^d L_P(-t) =& (-1)^d \frac{1}{d}(\mathrm{B}_d(-t+2) - \mathrm{B}_d) \\ =& (-1)^d \frac{1}{d}(\mathrm{B}_d(1-(t-1) - \mathrm{B}_d) \\ \overset{\text{lemmas}}{\underset{7.12,\ 7.13}{=}}& \frac{1}{d}(\mathrm{B}_d(t-1) - \mathrm{B}_d) = L_{P^\circ}(t)\end{aligned}$$

□

Next, we study the Ehrhart series. For this, we can even be more general: for $Q = \mathrm{Conv}(v_1, \ldots, v_m) \subset \mathbb{R}^{d-1}$ let $P = \mathrm{Pyr}(Q)$ be the pyramid

$$P = \mathrm{Conv}((\mathrm{v_1}, 0), \ldots, (\mathrm{v_m}, 0), \mathrm{e_d}) \subset \mathbb{R}^{\mathrm{d}}.$$

Then

$$L_{\mathrm{Pyr}(Q)}(t) = 1 + L_Q(1) + \ldots + L_Q(t).$$

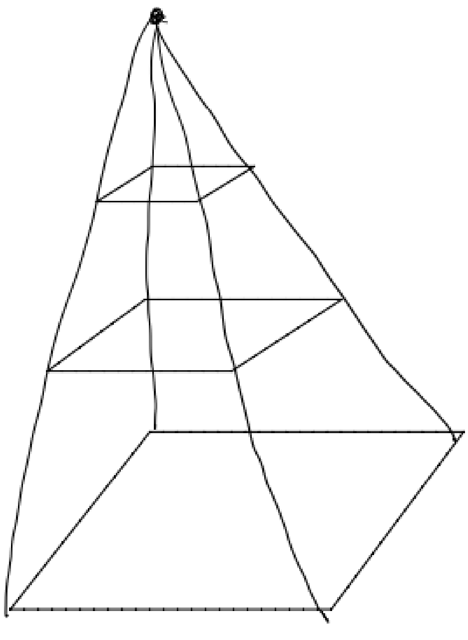


Abbildung 110: $L_{\mathrm{Pyr}(Q)(t)}$ [114]

7.15 Theorem.

$$\mathrm{Ehr}_{\mathrm{Pyr}(Q)}(z) = \frac{\mathrm{Ehr}_Q(z)}{1-z}$$

[114] Image from Hannah Markwig.

Beweis.

$$\begin{aligned}
\mathrm{Ehr}_{\mathrm{Pyr}(Q)}(z) &= 1 + \sum_{t\geq 1} L_{\mathrm{Pyr}(Q)}(t) z^t \\
&= 1 + \sum_{t\geq 1}\left(1 + \sum_{j=1}^{t} L_Q(j)\right) z^t \\
&= \sum_{t\geq 0} z^t + \sum_{t\geq 1}\sum_{j=1}^{t} L_Q(j) z^t \\
&= \frac{1}{1-z} + \sum_{j\geq 1} L_Q(j) \sum_{t\geq j} z^t \\
&= \frac{1}{1-z} + \sum_{j\geq 1} L_Q(j) \frac{z^j}{1-z} \\
&= \frac{1 + \sum_{j\geq 1} L_Q(j) z^j}{1-z} \\
&= \frac{\mathrm{Ehr}_Q(z)}{1-z}
\end{aligned}$$

□

With this, we can sum up our results for the pyramid P over the cube of dim $d-1$:

7.16 Theorem. *Let P be the pyramid over the cube of dim $d-1$.*

(1) Let lattice point count equals

$$L_P(t) = \frac{1}{d}(\mathrm{B}_d(t+2) - \mathrm{B}_d)$$

is a polynomial in t of deg d.

(2)

$$(-1)^d L_P(-t) = L_{P^\circ}(t)$$

(3)

$$\mathrm{Ehr}_P(z) = \frac{\sum_{k=1}^{d-1} A(d-1,k) z^{k-1}}{(1-z)^{d+1}}$$

Our next case study are <u>diamonds</u> (see figure 111):

$$P = \{(x_1, \ldots, x_d) \in \mathbb{R}^d \mid |x_1| + \ldots + |x_d| \leq 1\}$$

7.17 Definition. For $Q = \mathrm{Conv}(\mathrm{v}_1, \ldots, \mathrm{v}_\mathrm{m}) \subset \mathbb{R}^{\mathrm{d}-1}$ we define the **bipyramide**

$$P = \mathrm{bipyr}(Q) = \mathrm{Conv}((v_1, 0), \ldots, (v_m, 0), e_d, -e_d)$$

[115]Image from Hannah Markwig.

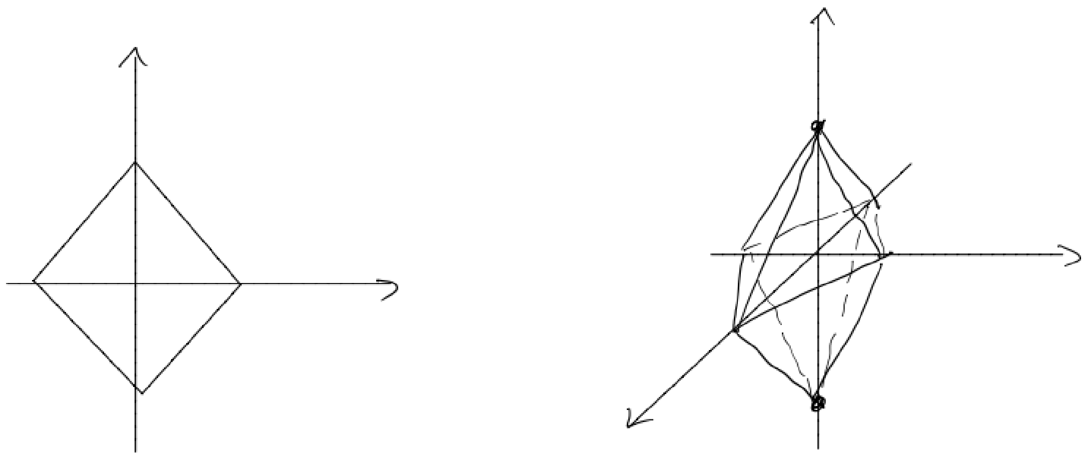


Abbildung 111: diamond [115]

7.18 Lemma.

$$L_{\mathrm{bipyr}(Q)}(t) = 2 + 2L_Q(1) + \ldots + 2L_Q(t-1) + L_Q(t) = 2 + 2\sum_{j=1}^{t-1} L_Q(j) + L_Q(t)$$

7.19 Theorem. *Assume* $0 \in Q$. *Then*

$$\mathrm{Ehr}_{\mathrm{bipyr(Q)}}(z) = \frac{1+z}{1-z} + \mathrm{Ehr}_Q(z).$$

Beweis.

$$\begin{aligned}
\mathrm{Ehr}_{\mathrm{bipyr(Q)}}(z) &= 1 + \sum_{t\geq 1} L_{\mathrm{bipyr}(Q)}(t) z^t \\
&= 1 + \sum_{t\geq 1}\Big(2 + 2\sum_{j=1}^{t-1} L_Q(j) + L_Q(t)\Big) z^t \\
&= 1 + 2\sum_{t\geq 1} z^t + 2\sum_{t\geq 1}\sum_{j=1}^{t-1} L_Q(j) z^t + \sum_{t\geq 1} L_Q(t) z^t \\
&= \mathrm{Ehr}_Q(z) + 2\Big(\sum_{t\geq 1} z^t + \sum_{t\geq 1}\sum_{j=0}^{t-1} L_Q(j) z^t\Big) \\
&= \mathrm{Ehr}_Q(z) + 2\cdot z\cdot\Big(\sum_{t\geq 0} z^t + \sum_{t\geq 0}\sum_{j=0}^{t} L_Q(j) z^t\Big) \\
&= \mathrm{Ehr}_Q(z) + 2\cdot z\cdot\Big(\frac{1}{1-z} + \sum_{j\geq 0} L_Q(j)\sum_{t\geq j} z^t\Big) \\
&= \mathrm{Ehr}_Q(z) + 2\cdot z\cdot\Big(\frac{1}{1-z} + \sum_{j\geq 0} L_Q(j)\frac{z^j}{1-z}\Big) \\
&= \mathrm{Ehr}_Q(z) + \frac{2z}{1-z}\mathrm{Ehr}_Q(z) \\
&= \frac{(1-z)\mathrm{Ehr}_Q(z) + 2z\mathrm{Ehr}_Q(z)}{1-z} \\
&= \mathrm{Ehr}_Q(z)\cdot\frac{1-z+2z}{1-z} = \mathrm{Ehr}_Q(z)\cdot\frac{1+z}{1-z}
\end{aligned}$$

□

With this, we can recursively determine the Ehrhart series of diamonds. For $d = 0$, the diamond is the 0-pt, with Ehrhart series

$$1 + \sum_{t\geq 1} 1 \cdot z^t = \sum_{t\geq 0} z^t = \frac{1}{1-z}$$

7.20 Proposition. *The Ehrhart series of the diamond in dim d is*

$$\mathrm{Ehr}_P(z) = \frac{(1+z)^d}{(1-z)^{d+1}}$$

7.21 Proposition. *The lattice point count of the diamond is*

$$L_P(t) = \sum_{k=0}^{d} \binom{d}{k}\binom{t-k+d}{d}$$

Beweis.

$$\begin{aligned}
\mathrm{Ehr}_P(t) =& \frac{(z+1)^d}{(1-z)^{d+1}} = \frac{\sum_{k=0}^d \binom{d}{k} z^k}{(1-z)^{d+1}} \\
=& \sum_{k=0}^{d} \binom{d}{k} z^k \cdot \Big(\sum_{t\geq 0} \binom{d+t}{d} z^t\Big) \\
=& \sum_{k=0}^{d} \binom{d}{k} \sum_{t\geq k} \binom{t-k+d}{d} z^t \\
\overset{(\star)}{=}& \sum_{k=0}^{d} \binom{d}{k} \sum_{t\geq 0} \binom{t-k+d}{d} z^t \\
=& 1 + \sum_{t\geq 0} L_P(t) z^t
\end{aligned}$$

with $(\star)$: since $\binom{t-k+d}{d} = 0$ for $0 \leq t < k$.
The claim follows by comparing coefficients. □

We sum up the results:

7.22 Theorem. *Let P be the diamond in dim d. Then:*

(1) The lattice path count is

$$L_P(t) = \sum_{k=0}^{d} \binom{d}{k}\binom{t-k+d}{d}$$

(2)

$$(-1)^d L_P(-t) = L_{P^\circ}(t)$$

(3)

$$\mathrm{Ehr}_P(z) = \frac{(1+z)^d}{(1-z)^{d+1}}$$

Beweis. 1), 3) was shown already in the propositions 7.20 and 7.21.

2)

$$\begin{aligned}
L_{P^\circ}(t) &= \#\{(m_1,\ldots,m_d) \in \mathbb{Z}^d \,|\, |m_1|+\ldots+|m_d| < t\} \\
&= \#\{(m_1,\ldots,m_d) \in \mathbb{Z}^d \,|\, |m_1|+\ldots+|m_d| \le t\} = L_P(t-1) \\
&\overset{1)}{=} \sum_{k=0}^{d} \binom{d}{k}\binom{t-1+d-k}{d} = \sum_{k=0}^{d} \binom{d}{d-k}\binom{t-1+d-k}{d} \\
&= \sum_{k=0}^{d} \binom{d}{k}\binom{t-1+k}{d}
\end{aligned}$$

(where we relabel with $k' = d-k$, and then rename k' as k). Since

$$(-1)^d\binom{d-t}{d} = \binom{t-1}{d}$$

(see standard simplex: lattice path count and reciprocity, where we also established $L_{\triangle^\circ}(t) = \binom{t-1}{d}$ for the interior lattice point count), this equals

$$(-1)^d \sum_{k=0}^{d} \binom{d}{k}\binom{d-k-t}{d} = (-1)^d L_P(-t)$$

□

As final case study, we consider polygons.

7.23 Theorem (Pick's formula)**.** *For a lattice polygon with i interior lattice points, b lattice points on the boundary and of area A, we have*

$$A = i + \frac{b}{2} - 1$$

(We have proved this formula already.)

7.24 Theorem. *Let P be a lattice polygon.*

(1) The lattice point count is

$$L_P(t) = At^2 + \frac{1}{2}bt + 1$$

(2)

$$L_P(-t) = L_{P^\circ}(t)$$

(3)

$$\mathrm{Ehr}_P(z) = \frac{(A - \frac{b}{2} + 1)z^2 + (A + \frac{b}{2} - 2)z + 1}{(1-z)^3}$$

Beweis. (1) The area of tP is At^2, the points on the boundary of tP are tb.

$$\begin{aligned} L_P(t) &= \text{inside pts } + \text{ boundary points of } tP \\ &= \text{inside pts } + tb \\ &\overset{\text{Pick}}{=} \mathrm{Area}(tP) - \frac{tb}{2} + 1 + tb \\ &= At^2 + \frac{b}{2}t + 1 \end{aligned}$$

(2)

$$L_P(-t) = At^2 - \frac{1}{2}bt + 1 \overset{\text{Pick}}{=} \text{ pts inside } tP = L_{P^\circ}(t)$$

(3)

$$\begin{aligned} \mathrm{Ehr}_P(z) &= 1 + \sum_{t\geq 1} L_P(t)z^t \\ &= 1 + \sum_{t\geq 1}(At^2 + \frac{b}{2}t + 1)z^t \\ &= 1 + A\sum_{t\geq 1} t^2z^t + \frac{b}{2}\sum_{t\geq 1} tz^t + \sum_{t\geq 1} z^t \\ &= A\sum_{t\geq 0} t^2z^t + \frac{b}{2}\sum_{t\geq 0} tz^t + \sum_{t\geq 0} z^t \\ &= A\frac{z+z^2}{(1-z)^3} + \frac{b}{2}\frac{z}{(1-z)^2} + \frac{1}{1-z} \\ &= \frac{Az + Az^2 + \frac{b}{2}(1-z)z + (1-z)^2}{(1-z)^3} \\ &= \frac{(A - \frac{b}{2} + 1)z^2 + (A + \frac{b}{2} - 2)z + 1}{(1-z)^3} \end{aligned}$$

□

After all these case studies revealing the same pattern, we aim for general results. We use the following result:

7.25 Theorem. *Let P be a polytope. Then there exists a triangulation of P whose vertices are contained in the vertices of P.*

Beweis. Let $A =$ vertices of P, we consider (A, P) as marked polytope. A point in the interior of a top-dimensional cone of the secondary fan of (A, P) yields such a triangulation. □

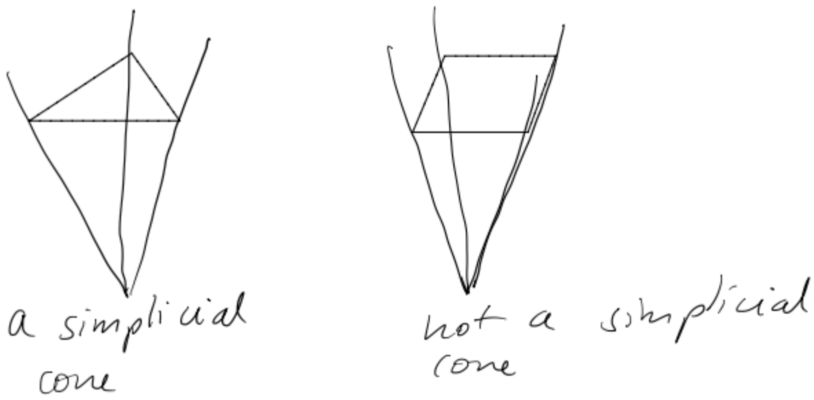


Abbildung 112: (not a) simplicial cone [116]

7.26 Definition. A cone is called **simplicial** if it is the cone over a simplex, i.e. if it has d ray generators for a cone of dim d.

7.27 Example. See figure 112.

In dim 2, every cone is simplicial.

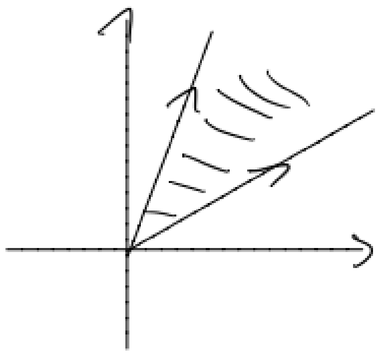


Abbildung 113: in dim 2 [117]

7.28 Definition. Let σ be a strictly convex cone. A **triangulation** of σ is a set T of simplicial cones s.th.

(1)

$$\sigma = \bigcup_{\sigma_i \in T} \sigma_i$$

(2) For $\sigma_1, \sigma_2 \in T$, the intersection $\sigma_1 \cap \sigma_2$ is a face of both.

7.29 Remark. If a polytope P is triangulated, s.th. cones over the simplices yield a triangulation of the cone over P.

7.30 Example. See figure 114.

[116]Image from Hannah Markwig.
[117]Image from Hannah Markwig.
[118]Image from Hannah Markwig.

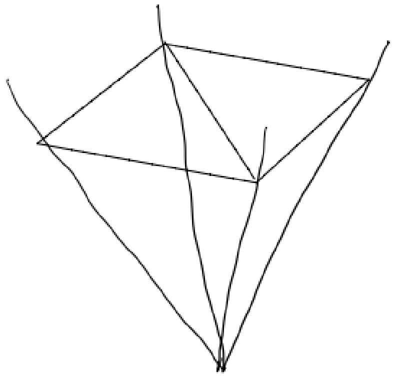


Abbildung 114: P with cones [118]

7.31 Definition. We say that σ is triangulated **without new generators**, if any ray generator of a cone in the triangulation is already a ray generator of σ.

By taking cones over a triangulation of a polytope P without new vertices, we obtain a triangulation of the cone over P without new generators. Thus, the following is a direct consequence of theorem 7.25:

7.32 Corollary. *For any strictly convex cone, there exists a triangulation without new generators.*

7.33 Definition. Let σ be a strictly convex, rational cone $\subset \mathbb{R}^d$. The **generating function** of σ is defined to be

$$f_\sigma(z) = f_\sigma(z_1, \dots, z_d) := \sum_{m \in \sigma \cap \mathbb{Z}^d} z^m = \sum_{\substack{m=(m_1,\dots,m_d) \\ \in \mathbb{Z}^d \cap \sigma}} z_1^{m_1} \cdot \ldots \cdot z_d^{m_d}$$

7.34 Example. $\sigma = [0, \infty) \subset \mathbb{R}$, $f_\sigma(z) = \sum_{m \in \mathbb{Z}_{\geq 0}} z^m = \frac{1}{1-z}$

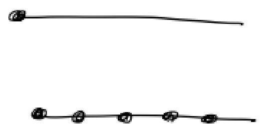
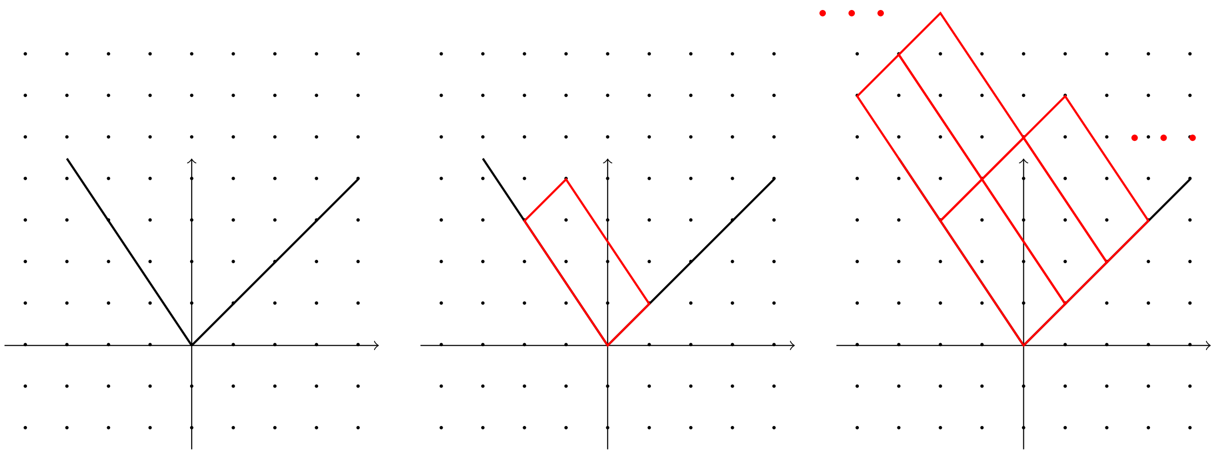


Abbildung 115: $f_\sigma(z)$ of σ [119]

7.35 Example. $\sigma = \mathrm{Cone}\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \\ 3 \end{pmatrix} \right) \subset \mathbb{R}^2$
We tile σ with shifts of the **fundamental parallelogram**

$$\Pi := \{\lambda_1(1,1) + \lambda_2(-2,3) \mid 0 \leq \lambda_1, \lambda_2 < 1\}$$

[119] Image from Hannah Markwig.

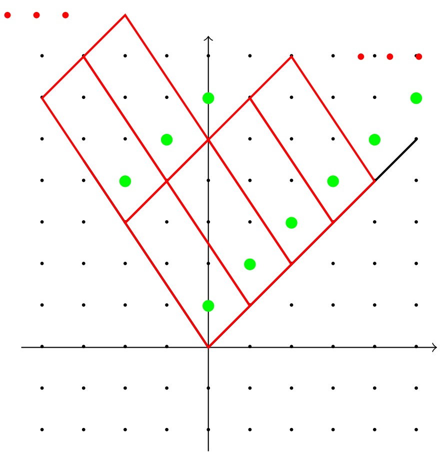


To understand the generating series of σ, we first study nonnegative integer multiples of the ray generators of σ:

$$\begin{aligned}\sum_{\substack{m=j(1,1)+k(-2,3)\\ j,k\geq 0}} z^m &= \sum_{j\geq 0}\sum_{k\geq 0} z^{j(1,1)+k(-2,3)}\\ &= \sum_{j\geq 0}\sum_{k\geq 0} z_1^{j-2k} z_2^{j+3k}\\ &= \sum_{j\geq 0}\sum_{k\geq 0} (z_1 z_2)^j \cdot (z_1^{-2} z_2^3)^k\\ &= \sum_{j\geq 0}(z_1 z_2)^j \cdot \sum_{k\geq 0}(z_1^{-2} z_2^3)^k\\ &= \frac{1}{1-z_1 z_2}\cdot\frac{1}{1-z_1^{-2} z_2^3}.\end{aligned}$$

For a given point $(m,n) \in \mathit{\Pi}$, we define

$$\mathcal{L}_{(m,n)} := \{(m,n) + j(1,1) + k(-2,3) \mid j,k \in \mathbb{Z}_{\geq 0}\}$$

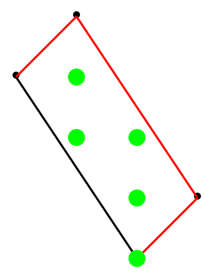


Then

$$\sigma \cap \mathbb{Z}^2 = \dot{\bigcup}_{(m,n)\in \mathit{\Pi}\cap\mathbb{Z}^2} \mathcal{L}_{(m,n)}$$

where

$$\mathit{\Pi} \cap \mathbb{Z}^2 = \{(0,0),(0,1),(0,2),(-1,2),(-1,3)\}$$



Thus we can write the generating series of σ as:

$$\begin{aligned}
f_\sigma(z_1,z_2) &= \sum_{p\in\mathbb{Z}^2\cap\sigma} z_1^{p_1} z_2^{p_2} \\
&\overset{(*)}{=} \sum_{(m,n)\in\mathit{\Pi}\cap\mathbb{Z}^2} \sum_{p\in\mathcal{L}_{(m,n)}} z_1^{p_1} z_2^{p_2} \\
&= \sum_{(m,n)\in\mathit{\Pi}\cap\mathbb{Z}^2} z_1^m z_2^n \cdot \Bigg(\sum_{\substack{j(1,1)+k(-2,3)\\ j,k\geq 0}} z^{j(1,1)+k(-2,3)} \Bigg) \\
&= \sum_{(m,n)\in\mathit{\Pi}\cap\mathbb{Z}^2} z_1^m z_2^n \cdot \frac{1}{1-z_1z_2} \cdot \frac{1}{1-z_1^{-2}z_2^3} \\
&= \left(1+z_2+z_2^2+z_1^{-1}z_2^2+z_1^{-1}z_2^3\right) \cdot \frac{1}{1-z_1z_2} \cdot \frac{1}{1-z_1^{-2}z_2^3}.
\end{aligned}$$

With $(*) : p = (m,n) + j(1,1) + k(-2,3),\ j,k \in \mathbb{Z}_{\geq 0}$

7.36 Theorem. *Let $\sigma = \operatorname{Cone}(w_1,\ldots,w_d) \subset \mathbb{R}^d$ be a simplicial, (strictly convex), rational cone. Consider for $v \in \mathbb{R}^d$ the shift $v+\sigma$. The generating function of the shift is*

$$f_{v+\sigma}(z) = \frac{f_{v+\mathit{\Pi}}(z)}{(1-z^{w_1})\cdot\ldots\cdot(1-z^{w_d})}$$

where $\mathit{\Pi}$ is the partially open parallelogram

$$\mathit{\Pi} = \{\lambda_1 w_1 + \ldots + \lambda_d w_d | 0 \leq \lambda_i < 1\}$$

$\mathit{\Pi}$ is called the ***fundamental parallelogram****.*

Beweis. A lattice point $m \in (v+\sigma)\cap\mathbb{Z}^d$ (contributing a summand z^m to f_σ) can be written as

$$m = v + \lambda_1 w_1 + \ldots + \lambda_d w_d$$

with $\lambda_i \in \mathbb{R}_{\geq 0}$
Since the w_i form a basis of $\mathbb{R}^d$, the λ_i above are unique. We write each λ_i as sum of integer part and decimal:

$$\lambda_i = \lfloor\lambda_i\rfloor + \{\lambda_i\}, \quad \lfloor\lambda_i\rfloor \in \mathbb{Z},\ 0 \leq \{\lambda_i\} < 1.$$

Then

$$m = v + (\lfloor \lambda_1 \rfloor + \{\lambda_1\})w_1 + \ldots + (\lfloor \lambda_d \rfloor + \{\lambda_d\})w_d$$

and

$$p := v + \{\lambda_1\}w_1 + \ldots + \{\lambda_d\}w_d \in v + \mathit{\Pi}$$

Furthermore, since $m \in \mathbb{Z}^d$ and $\lfloor \lambda_1 \rfloor w_1 + \ldots + \lfloor \lambda_d \rfloor w_d \in \mathbb{Z}^d$, also $p \in \mathbb{Z}^d$. Hence, every $m \in (v + \sigma) \cap \mathbb{Z}^d$ can uniquely be written as

$$m = p + k_1 w_1 + \ldots + k_d w_d$$

for $p \in (v + \mathit{\Pi}) \cap \mathbb{Z}^d$ and $k_i \in \mathbb{Z}_{\geq 0}$. The function

$$\begin{aligned} \frac{f_{v+\mathit{\Pi}}(z)}{(1-z^{w_1}) \cdot \ldots \cdot (1-z^{w_d})} &= f_{v+\mathit{\Pi}}(z) \cdot \Big(\sum_{k_1 \geq 0} z^{k_1 w_1}\Big) \cdot \ldots \cdot \Big(\sum_{k_d \geq 0} z^{k_d w_d}\Big) \\ &= \sum_{p \in (v+\mathit{\Pi}) \cap \mathbb{Z}^d} z^p \cdot \Big(\sum_{k_1 \geq 0} z^{k_1 w_1}\Big) \cdot \ldots \cdot \Big(\sum_{k_d \geq 0} z^{k_d w_d}\Big) \end{aligned}$$

If we multiply out, the exponent of a monomial in this expression exactly takes such a form. The equality

$$f_{v+\sigma}(z) = \frac{f_{v+\mathit{\Pi}}(z)}{(1-z^{w_1}) \cdot \ldots \cdot (1-z^{w_d})}$$

follows. □

7.37 Corollary. *Let $\sigma = \mathrm{Cone}(w_1, \ldots, w_d) \subset \mathbb{R}^d$ be a simplicial, (strictly convex), rational cone. Consider for $v \in \mathbb{R}^d$ the shift $v + \sigma$. Assume there are no lattice points in the boundary of $v + \sigma$. The generating function of $v + \sigma$ is*

$$f_{v+\sigma}(z) = \frac{f_{v+\mathit{\Pi}}(z)}{(1-z^{w_1}) \cdot \ldots \cdot (1-z^{w_d})}$$

where $\mathit{\Pi}$ is the open parallelogram

$$\mathit{\Pi} = \{\lambda_1 w_1 + \ldots + \lambda_d w_d | 0 \leq \lambda_i < 1\}.$$

Beweis. The proof follows the same ideas as for theorem 7.36. Since there are no lattice points on the boundary we can safely work with the open parallelogram $\mathit{\Pi}$. □

7.38 Corollary. *The generating function of any strict convex cone is a rational function in the z_i.*

Beweis. This follows since we can use a triangulation and theorem 7.36. If we insert new facets in the interior, their lattice points would be overcounted if we just add contributions from the simplicial cones. So we have to subtract such contributions again. But faces of simplical cones are always simplicial, so the contributions we subtract are again rational functions. If we subtract contributions from new facets, it is possible that we subtract too much: new codim-2-faces that are faces of several facets. We have to add their contributions again. Overall, we have to use an inclusion-exclusion principle, where we add (resp. subtract) rational functions. □

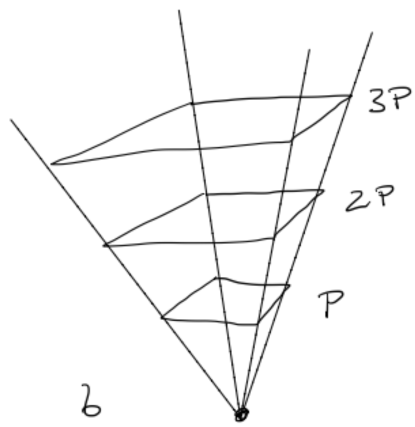


Abbildung 116: σ [120]

7.39 Remark. We can use the previous results for the lattice point count of polytopes. Assume σ is the cone over a polytope $P \subset \mathbb{R}^d$. See 116

Then

$$f_\sigma(1, \ldots, 1, z_{d+1}) = 1 + \sum_{t \geq 1} \#(tP \cap \mathbb{Z}^d) z_{d+1}^t = 1 + \sum_{t \geq 1} L_P(t) z_{d+1}^t = \mathrm{Ehr}_P(z_{d+1})$$

The following technical lemma will be useful for the further study of lattice point counts and Ehrhart series:

7.40 Lemma. *If a generating series*

$$\sum_{t \geq 0} f(t) z^t = \frac{g(z)}{(1-z)^{d+1}},$$

then f is a polynomial of degree d $\Leftrightarrow$ g is a polynomial of degree at most d and $g(1) \neq 0$.

Beweis. "$\Rightarrow$ " Assume $f = a_d x^d + \ldots + a_0$, where $a_d \neq 0$. Then

$$\begin{aligned}
\sum_{t \geq 0} f(t) z^t &= \sum_{t \geq 0} (a_d t^d + \cdots + a_0) z^t \\
&= a_d \sum_{t \geq 0} t^d z^t + a_{d-1} \sum_{t \geq 0} t^{d-1} z^t + \ldots + a_0 \sum_{t \geq 0} z^t. \\
&= a_d \cdot \frac{\sum_{k=0}^d A(d,k) z^k}{(1-z)^{d+1}} + a_{d-1} \frac{\sum_{k=0}^{d-1} A(d-1,k) z^k}{(1-z)^d} + \cdots + a_0 \cdot \frac{1}{1-z} \\
&= \left(a_d \Big(\sum_{k=0}^d A(d,k) z^k \Big) + a_{d-1} \Big(\sum_{k=0}^{d-1} A(d-1,k) z^k \Big) (1-z) + \cdots + a_0 (1-z)^d \right) \cdot \frac{1}{(1-z)^{d+1}}
\end{aligned}$$

So we have

$$g = a_d \left(\sum_{k=0}^d A(d,k) z^k \right) + a_{d-1} \left(\sum_{k=0}^{d-1} A(d-1,k) z^k \right) (1-z) + \cdots + a_0 (1-z)^d.$$

[120] Image from Hannah Markwig.

It is a polynomial of degree at most d, and

$$g(1) = a_d \left(\sum_{k=0}^{d} A(d,k) \right) = a_d d! \neq 0.$$

"$\Leftarrow$"

$$\frac{g(z)}{(1-z)^{d+1}} = g(z) \cdot \sum_{t \geq 0} \binom{d+t}{d} z^t.$$

Assume $g(z) = b_d z^d + \cdots + b_0$, then

$$\begin{aligned}
\frac{g(z)}{(1-z)^{d+1}} =& (b_d z^d + \cdots + b_0) \cdot \sum_{t\geq 0} \binom{d+t}{d} z^t \\
=& b_d \cdot \sum_{t \geq 0} \binom{d+t}{d} z^{t+d} + b_{d-1} \cdot \sum_{t\geq 0} \binom{d+t}{d} z^{t+d-1} + \ldots + b_0 \sum_{t \geq 0} \binom{d+t}{d} z^t \\
\overset{(*)}{=} & \sum_{k \geq 0} \left(b_d \binom{k}{d} + b_{d-1} \binom{k+1}{d} + \cdots + b_0 \binom{k+d}{d} \right) z^k.
\end{aligned}$$

With $(*)$: sort for $z^k, (k = t+d \;\Rightarrow\; t = k-d)$ or $(k = t+d-1 \;\Rightarrow\; t = k-d+1, \ldots)$
Thus, the polynomial $f(X)$ is

$$\begin{aligned}
f(X) &= b_d \binom{X}{d} + b_{d-1} \binom{X+1}{d} + \ldots + b_0 \binom{X+d}{d}. \\
&= b_d \frac{1}{d!} X(X-1) \cdot \ldots \cdot (X-d+1) + \\
&+ b_{d-1} \frac{1}{d!} (X+1) X (X-1) \cdot \ldots \cdot (X-d+2) + \ldots + \\
&+ b_0 \frac{1}{d!} (X+d)(X+d-1) \cdot \ldots \cdot (X+1).
\end{aligned}$$

The X^d-coefficient of $f(X)$ equals

$$(b_d + b_{d-1} + \ldots + b_0) \cdot \frac{1}{d!} = g(1) \cdot \frac{1}{d!} \neq 0$$

$\Rightarrow f$ is a polynomial of degree d. $\square$

7.41 Theorem (Ehrhart)**.** *If P is a lattice polytope of dimension d, then $L_P(t)$ is a polynomial in t of degree d.*

Beweis. It is sufficient to prove theorem for simplices, since we can triangulate any lattice polytope. The simplices in a triangulation intersect only in lower-dimensional simplices, so any correction term we have to add/subtract with an inclusion or exclusion principle does not effect the top degree part.
Notice that the leading coefficient of any $L_Q(t)$ for a simplex Q must be positive, since $\lim_{t\to\infty} L_Q(t) = \infty$.
Hence adding $L_Q(t)$ for d-dimensional simplices in a triangulation of P, (and adding/subtracting

correction terms of lower degree), we obtain a polynomial of degree d for $L_P(t)$.
It remains to prove the theorem for simplices.
Let $\triangle$ be a d-dimensional lattice simplex. Because of lemma 7.40, it is sufficient to show

$$\mathrm{Ehr}_\triangle(z) = 1 + \sum_{t\geq 1} L_\triangle(t) z^t = \frac{g(z)}{(1-z)^{d+1}}$$

for a polynomial g of degree at most d satisfying $g(1) \neq 0$. We have

$$\mathrm{Ehr}_\triangle(z) = f_\sigma(1,\dots,1,z),$$

where σ is the cone over $\triangle$ and f_σ its generating series. By theorem 7.36,

$$f_\sigma(z_1,\dots,z_{d+1}) = \frac{f_\Pi(z,\dots,z_{d+1})}{(1-z^{w_1})\cdot\ldots\cdot(1-z^{w_{d+1}})}$$

where the $w_i = (v_i, 1)$ generate σ ($\triangle = \mathrm{Conv(v_i)}$) and

$$\Pi = \{\lambda_1 w_1 + \ldots + \lambda_{d+1} w_{d+1} : 0 \leq \lambda_i < 1\}$$

is the fundamental parallelogram. Since Π is bounded, it contains finitely many lattice points and $f_\Pi(z_1,\dots,z_{d+1})$ is a Laurent polynomial in the z_i.

<u>Claim:</u> $\deg_{z_{d+1}}(f_\Pi) \leq d$. The X_{d+1}-coordinate of every w_i is 1, hence the X_{d+1}-coordinate of a point in Π is

$$\lambda_1 + \ldots + \lambda_{d+1}$$

for some $0 \leq \lambda_i < 1 \Rightarrow$ the X_{d+1}-coordinate $< d+1$, for a lattice point, $\leq d$.
$\Rightarrow f_\Pi(1,\dots,1,z_{d+1})$ is a polynomial of degree at most d in z_{d+1}. Furthermore,

$$f_\Pi(1,\dots,1) = \#(\Pi \cap \mathbb{Z}^d) \neq 0.$$

It follows that

$$\begin{aligned}
\mathrm{Ehr}_\triangle(z) =& f_\sigma(1,\dots,1,z)\\
=& \left.\frac{f_\Pi(z_1,\dots,z_{d+1})}{(1-z^{w_1})\cdot\ldots\cdot(1-z^{w_{d+1}})}\right|_{z_1=\ldots=z_d=1,\, z_{d+1}=z}\\
\overset{(*)}{=}& \frac{f_\Pi(1,\dots,1,z)}{(1-z)\cdots(1-z)}\\
=& \frac{f_\Pi(1,\dots,1,z)}{(1-z)^{d+1}}.
\end{aligned}$$

With $(*)$: Since $1 - z^{w_i} = 1 - z_1^{v_{i_1}} \ldots z_d^{v_{i_d}} z_{d+1}$
The claim follows. $\square$

7.42 Corollary. *The Ehrhart series of a polytope of dimension d takes the form*

$$\frac{g(z)}{(1-z)^{d+1}}$$

for some polynomial g.

We will study this polynomial g further. The following is a direct consequence of the previous proof:

7.43 Corollary. *Let $\triangle$ be a lattice simplex of dimension d with vertices $v_1, \dots, v_{d+1}$, let $w_j = (v_j, 1)$. Then*

$$\mathrm{Ehr}_\triangle(z) = \frac{h_d z^d + h_{d-1}z^{d-1} + \ldots + h_1 z + h_0}{(1-z)^{d+1}}$$

where $h_k = \#$ lattice points in $\{\lambda_1 w_1 + \ldots + \lambda_{d+1} w_{d+1} : 0 \leq \lambda_i < 1\}$ whose last coordinate is k.

7.44 Theorem (Stanley's nonnegativity Theorem)**.** *Let P be a d-dimensional lattice polytope with*

$$\mathrm{Ehr}_P(z) = \frac{h_d z^d + \ldots + h_0}{(1-z)^{d+1}}$$

Then $h_0, \dots, h_d \geq 0$.

Beweis. We triangulate $\sigma = \mathrm{Cone}(P)$.
Claim: $\exists\, v \in \mathbb{R}^{d+1}$ s.th. the facets of $v + \sigma$ contain no lattice points, nor do any of the facets of the simplices in its triangulation.
For a single rational hyperplane

$$H = \{x \in \mathbb{R}^d | a_1 x_1 + \ldots + a_d x_d = 0\}$$

with $a_i \in \mathbb{Z}$, we can use theorem 5.17 to find a lattice basis B of $H \cap \mathbb{Z}^d$ and another vector $v \in \mathbb{Z}^d$, s.th. $B \cup \{v\}$ is a basis of $\mathbb{Z}^d$. If shift H using integer multiples of v, we cover all the lattice points. If we shift H by λv with $\lambda \notin \mathbb{Z}$, we cannot cover any lattice point. Adding vectors in H to λv does not change the shift.
We can do this for any hyperplane containing the facets of the simplices of our triangulation. We have enough choices for each so that we can pick a vector with which we can shift all while avoiding lattice points.

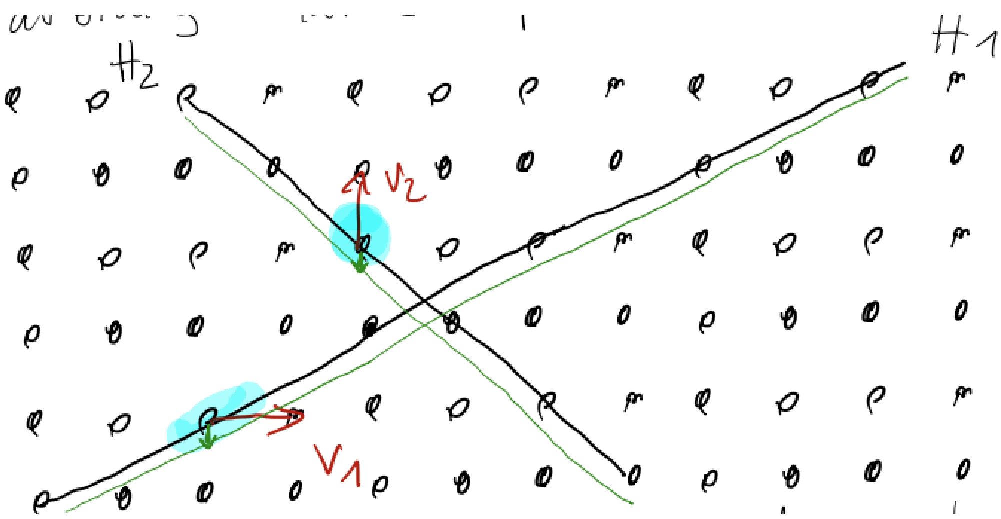


Abbildung 117: H_1, H_2, v_1 and v_2 [121]

In figure 117, we pick v_1 and v_2 to add to the lattice basis. Suitable shifts can be found in a $||\varepsilon v_i||$-neighbourhood, the green vector is in the union of the two neighbourhoods, the green shifts do not contain lattice points.
For the shifted cone, we have

$$\sigma \cap \mathbb{Z}^{d+1} = (v + \sigma) \cap \mathbb{Z}^{d+1},$$

and no lattice point lies on any facet of the triangulation, i.e.,

$$\begin{aligned}
&\sigma \cap \mathbb{Z}^{d+1} = (v + \sigma) \cap \mathbb{Z}^{d+1} = \dot{\bigcup}_{j=1}^{m} (v + \sigma_j) \cap \mathbb{Z}^{d+1} \\
\Rightarrow & f_\sigma(z_1, \dots, z_{d+1}) = \sum_{j=1}^{m} f_{\sigma_j}(z_1, \dots, z_{d+1}) \\
\Rightarrow & \mathrm{Ehr}_P(z) = f_\sigma(1, \dots, 1, z) = \sum_{j=1}^{m} f_{\sigma_j}(1, \dots, 1, z)
\end{aligned}$$

<u>Claim:</u> The rational function

$$f_{\sigma_i}(1, \dots, 1, z)$$

has nonnegative coefficients in its numerator.
This follows from the previous corollary, where we showed for such a coefficient

$$h_k = \#\text{ lattice points in } \{\lambda_1 w_1 + \dots + \lambda_{d+1} w_{d+1} \mid 0 \leq \lambda_i < 1\}$$

whose last coordinate is k. □

7.45 Corollary. *Let P be a d-dimensional lattice polytope with*

$$Ehr_P(z) = \frac{h_d z^d + \dots + h_0}{(1 - z)^{d+1}}.$$

Then $h_0 = 1$.

7.46 Proposition. *Let P be a d-dimensional lattice polytope with*

$$Ehr_P(z) = \frac{h_d z^d + \dots + 1}{(1 - z)^{d+1}}.$$

Then

$$L_P(t) = \binom{t+d}{d} + h_1 \binom{t+d-1}{d} + \dots + h_{d-1} \binom{t+1}{d} + h_d \binom{t}{d}.$$

Beweis.

$$\begin{aligned}
\mathrm{Ehr}_P(z) = & \left(h_d z^d + \dots + h_1 z + 1\right) \cdot \frac{1}{(1 - z)^{d+1}} \\
= & (h_d z^d + \dots + h_0) \cdot \sum_{t \geq 0} \binom{t+d}{d} z^t \\
= & h_d \sum_{t \geq 0} \binom{t+d}{d} z^{t+d} + h_{d-1} \sum_{t \geq 0} \binom{t+d}{d} z^{t+d-1} + \ldots + h_1 \sum_{t \geq 0} \binom{t+d}{d} z^{t+1} + \sum_{t \geq 0} \binom{t+d}{d} z^t \\
\overset{(*)}{=} & \sum_{t \geq 0} \left(h_d \binom{t}{d} + h_{d-1} \binom{t+1}{d} + \ldots + h_1 \binom{t+d-1}{d} \binom{t+d}{d} \right) z^t
\end{aligned}$$

[121]Image from Hannah Markwig.

With $(*)$: sort for z^k, $\quad k = t + d \Rightarrow t = k - d$ or $k = t + d - 1 \Rightarrow t = k - d + 1, \ldots$

$$L_P(t) = \binom{t+d}{d} + h_1\binom{t+d-1}{d} + \ldots + h_{d-1}\binom{t+1}{d} + h_d\binom{t}{d}.$$

□

7.47 Corollary. *The constant term of $L_P(t)$ is 1.*

Beweis. $L_P(0) = \binom{d}{d} = 1$ □

Accordingly, we can write the Ehrhart series as

$$\mathrm{Ehr}_P(z) = \sum_{t \geq 0} L_P(t) z^t.$$

7.48 Corollary. *Let p be a d-dimensional lattice polytope with*

$$\mathrm{Ehr}_P(z) = \frac{h_d z^d + \ldots + h_0}{(1-z)^{d+1}}.$$

Then

$$h_1 = L_P(1) - d - 1 = \# P \cap \mathbb{Z}^d - d - 1.$$

Beweis.

$$L_P(1) = \binom{1+d}{d} + h_1\binom{1+d-1}{d} + \cdots + h_{d-1}\binom{2}{d} + h_d\binom{1}{d} = d + 1 + h_1.$$

□

7.49 Corollary. *Let $L_P(t) = c_d t^d + \cdots + c_1 t + 1$, then*

$$d! \cdot c_i \in \mathbb{Z} \quad \forall i = 1, \ldots, d.$$

Beweis. Since

$$\begin{aligned} L_P(t) &= \binom{t+d}{d} + h_1\binom{t+d-1}{d} + \ldots + h_{d-1}\binom{t+1}{d} + h_d\binom{t}{d} \\ &= \frac{1}{d!}(t+d) \cdot \ldots \cdot (t+1) + h_1 \cdot \frac{1}{d!}(t+d-1) \cdot \ldots \cdot t + \ldots \end{aligned}$$

□

The following will be useful when we evaluate $L_P(t)$ for negative values.

7.50 Proposition. *Let p be a polynomial of degree d s.th.*

$$\sum_{t \geq 0} p(t) z^t = \frac{h_d z^d + h_{d-1} z^{d-1} + \cdots + h_1 z + h_0}{(1-z)^{d+1}}.$$

Then

$$h_d = h_{d-1} = \cdots = h_{k+1} = 0$$

and

$$h_k \neq 0 \quad \Leftrightarrow \quad p(-1) = p(-2) = \cdots = p(-(d+k)) = 0.$$

and

$$p(-(d-k+1)) \neq 0.$$

Beweis. "$\Rightarrow$" Let

$$h_d = h_{d-1} = \cdots = h_{k+1} = 0 \quad \text{and} \quad h_k \neq 0.$$

Then

$$\begin{aligned}(h_k z^k + \ldots + h_1 z + h_0) \cdot \frac{1}{(1-z)^{d+1}} =& (h_k z^k + \ldots + h_0) \cdot \sum_{t \geq 0} \binom{t+d}{d} z^t \\ =& h_k \sum_{t \geq 0} \binom{t+d}{d} z^{t+k} + h_{k-1} \sum_{t \geq 0} \binom{t+d}{d} z^{t+k-1} + \ldots \\ & + h_1 \sum_{t \geq 0} \binom{t+d}{d} z^{t+1} + \sum_{t \geq 0} \binom{t+d}{d} z^t \\ =& \sum_{t \geq 0} \left(h_k \binom{t+d-k}{d} + h_{k-1} \binom{t+d-k+1}{d} + \ldots \right. \\ & \left. + h_0 \binom{t+d}{d} \right) z^t\end{aligned}$$

$$\begin{aligned}\Rightarrow \; p(t) &= h_k \binom{t+d-k}{d} + h_{k-1} \binom{t+d-k+1}{d} + \cdots + h_0 \binom{t+d}{d} \\ &= h_k \frac{1}{d!} (t+d-k)(t+d-k-1) \cdot \ldots \cdot (t-k+1) \\ &+ h_{k-1} \frac{1}{d!} (t-d-k+1) \cdot \ldots \cdot (t-k+2) + \ldots \\ &+ h_0 \frac{1}{d!} (t+d) \cdot \ldots \cdot (t+1).\end{aligned}$$

All binomial coefficients are 0 for $t = -1, -2, \ldots, -d+k$.
For $t = -d+k-1$, all binomial coefficients except the first,

$$h_k \frac{1}{d!} (t+d-k)(t+d-k-1) \cdot \ldots \cdot (t-k+1),$$

which is nonzero since $h_k \neq 0$.

$$\Rightarrow p(-1) = \cdot \ldots \cdot = p(-(d-k)) = 0 \quad \text{and} \quad p(-(d-k+1)) \neq 0.$$

"$\Leftarrow$" As before, we can see that

$$\begin{aligned}p(t) &= h_d \binom{t}{d} + h_{d-1} \binom{t+1}{d} + \cdot \ldots \cdot + h_0 \binom{t+d}{d} \\ &= h_d \frac{1}{d!} t \cdot (t-1) \cdot \ldots \cdot (t-d+1) \\ &+ h_{d-1} \frac{1}{d!} (t+1) \cdot t \cdot \ldots \cdot (t-d+2) + \ldots \\ &+ h_0 \frac{1}{d!} (t+d) \cdot \ldots \cdot (t+1).\end{aligned}$$

Then,

$$0 = p(-1) = h_d \cdot \frac{1}{d!}(-1)(-2)\dots(-d) = h_d \cdot (-1)^d \quad \Rightarrow h_d = 0.$$
$$0 = p(-2) = h_{d-1}\frac{1}{d!}(-1)(-2)\dots(-d) = h_{d-1} \cdot (-1)^d \quad \Rightarrow h_{d-1} = 0.$$

Continuing, we can see that the fact that $p(-i) = 0$ implies

$$h_{d-i+1} = 0 \quad \text{for } i = 1, \dots, d-k. \quad \Rightarrow h_d = \dots = h_{k+1} = 0.$$

If furthermore $h_k = 0$, we could equivalently see $p(-(d-k+1)) = 0$ which contradicts our assumption. Hence $h_k \neq 0$. □

Counting lattice points in stretched polytopes is related to the computation of volume:

7.51 Remark. Let $S \subset \mathbb{R}^d$ be a subset of dim d, then

$$\text{vol } S = \lim_{t\to\infty} \frac{1}{t^d} \# \left(S \cap \left(\frac{1}{t}\mathbb{Z}\right)^d \right)$$

because we can think of filling up S with little boxes of side length $\frac{1}{t}$, which have volume $\frac{1}{t^d}$. The count of lattice points corresponds to the count of boxes.
Counting the lattice points of the lattice $\left(\frac{1}{t}\mathbb{Z}\right)^d$ is as good as counting the $\mathbb{Z}^d$-lattice points of tS. Thus

$$\text{vol } S = \lim_{t\to\infty} \frac{1}{t^d} \#(tS \cap \mathbb{Z}^d).$$

7.52 Example. Let $P \subset \mathbb{R}^2$ be a lattice polygon. We know that $L_P(t) = At^2 + \frac{1}{2}bt + 1$, where A is its area and $b = \#$ lattice points on the boundary. Then

$$\begin{aligned} \lim_{t\to\infty} \frac{1}{t^2} \#(tP \cap \mathbb{Z}^2) &= \lim_{t\to\infty} \frac{1}{t^2} L_P(t) \\ &= \lim_{t\to\infty} \frac{1}{t^2} \cdot \left(At^2 + \frac{1}{2}bt + 1 \right) \\ &= \lim_{t\to\infty} \left(A + \frac{b}{2t} + \frac{1}{t^2} \right) = A, \end{aligned}$$

which is the area, i.e., the volume, of P.

7.53 Proposition. *Let $P \subset \mathbb{R}^d$ be a d-dim lattice polytope with*

$$L_P(t) = c_d t^d + c_{d-1} t^{d-1} + \ldots + c_1 t + 1.$$

Then $c_d = \text{vol}(P)$.

Beweis.

$$\begin{aligned} \text{vol}(P) &= \lim_{t\to\infty} \frac{c_d t^d + c_{d-1} t^{d-1} + \ldots + c_1 t + 1}{t^d} \\ &= \lim_{t\to\infty} \left(c_d + \frac{c_{d-1}}{t} + \dots + \frac{c_1}{t^{d-1}} + \frac{1}{t^d} \right) \\ &= c_d. \end{aligned}$$

□

7.54 Corollary. *Let $P \subset \mathbb{R}^d$ be a d-dim lattice polytope with*

$$\mathrm{Ehr}_P(z) = \frac{h_d z^d + h_{d-1} z^{d-1} + \cdots + h_1 z + 1}{(1-z)^{d+1}}.$$

Then

$$\mathrm{vol}(P) = \frac{1}{d!}\left(h_d + h_{d-1} + \cdots + h_1 + 1\right).$$

Beweis. We know that

$$L_P(t) = \binom{t+d}{d} + h_1 \binom{t+d-1}{d} + \cdots + h_{d-1}\binom{t+1}{d} + h_d \binom{t}{d},$$

and the leading coefficient of this polynomial in t is the volume. The leading coefficient is

$$\frac{1}{d!}\left(1 + h_1 + \cdots + h_d\right).$$

□

Questions:

- What is the number of triangulations of a polytope without new vertices? (Or: What is the number of top-dim cones in the secondary fan $(P, \mathrm{vert}(P))$, or in any secondary fan?
- What is the minimal number of simplices needed to triangulate the cube $[0,1]^d$? (Known for $d \leq 7$).
- What are all polynomials of degree d which are Ehrhart-polynomials, i.e., of the form $L_P(t)$ for a polytope P of dimension d? (Known for $d = 2$, partial results for $d = 3, 4$).
- What can we say about the zeros of Ehrhart polynomials?
- Let P, Q be lattice polytopes with $L_P(t) = L_Q(t)$. What further requirements do P and Q need to satisfy to guarantee that $P = f(Q)$ for an affine map f whose linear part is $\mathbb{Z}$-invertible?

Our next goal is to prove

7.55 Theorem (Ehrhart-Macdonald Reciprocity)**.** *Let P be a lattice polytope. Then*

$$L_P(-t) = (-1)^{\dim P} L_{P^\circ}(t).$$

7.56 Remark. $L_P(t)$ is a function which, for $t \geq 0$, counts lattice points. For $t > 0$, $L_P(-t)$ a priori does not make sense (not in the original meaning).
But, given that $L_P(t)$ is a polynomial in t, we can still of course insert negative values. EM-Reciprocity gives a meaning to the Ehrhart polynomial $L_P(t)$ evaluated at negative values.

We first study simplicial cones.

7.57 Proposition. *Let $w_1, \dots, w_d \in \mathbb{Z}^d$ be linearly independent and let*

$$\sigma = \operatorname{Cone}(w_1, \dots, w_d).$$

Let $v \in \mathbb{R}^d$ such that the boundary of the shifted cone $v + \sigma$ contains no lattice points. Then

$$f_{v+\sigma}\left(\frac{1}{z_1}, \dots, \frac{1}{z_d}\right) = (-1)^d f_{-v+\sigma}(z_1, \dots, z_d)$$

as rational functions.

Beweis. We know that

$$f_{v+\sigma}(z_1, \dots, z_d) = \frac{f_{v+\Pi}(z_1, \dots, z_d)}{(1 - z^{w_1}) \cdot \ldots \cdot (1 - z^{w_d})},$$

where Π is the open fundamental parallelogram,

$$\Pi = \{\lambda_1 w_1 + \cdots + \lambda_d w_d \mid 0 < \lambda_i < 1\}.$$

Analogously,

$$f_{-v+\sigma}(z_1, \dots, z_d) = \frac{f_{-v+\Pi}(z_1, \dots, z_d)}{(1 - z^{w_1}) \cdot \ldots \cdot (1 - z^{w_d})}.$$

□

<u>Claim:</u> $v + \Pi = -(-v + \Pi) + w_1 + \ldots + w_d$

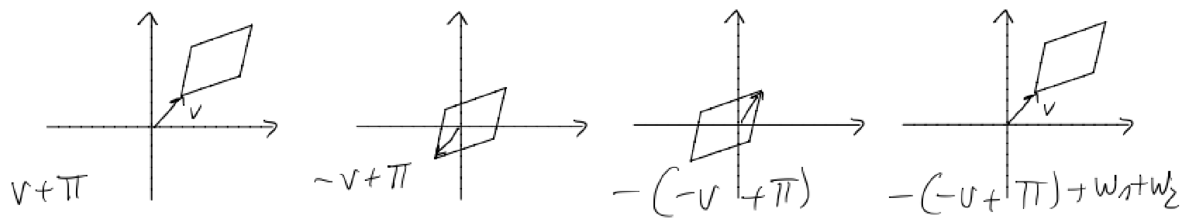


Abbildung 118: $v + \Pi$, $-v + \Pi$, $-(-v + \Pi)$, $-(-v + \Pi) + w_1 + w_2$ [122]

Let $p \in v + \Pi$

$$\Rightarrow \quad p = v + \lambda_1 w_1 + \ldots + \lambda_d w_d, \quad 0 < \lambda_i < 1.$$
$$p - w_1 - \ldots - w_d = v + (\lambda_1 - 1) w_1 + \ldots + (\lambda_d - 1) w_d.$$

Let

$$\begin{aligned}
&\mu_i := \lambda_i - 1 \text{ as } 0 < \lambda_i < 1 \\
&\Rightarrow -1 < \mu_i < 0 \\
&\Rightarrow 1 > -\mu_i > 0 \\
&\Rightarrow -\mu_1 w_1 - \ldots - \mu_d w_d \in \Pi \\
&\Rightarrow -v - \mu_1 w_1 - \ldots - \mu_d w_d \in -v + \Pi \\
&\Rightarrow v + \mu_1 w_1 + \ldots + \mu_d w_d \in -(-v + \Pi) \\
&\Rightarrow p - w_1 - \ldots - w_d \in -(-v + \Pi) \\
&\Rightarrow p \in -(-v + \Pi) + w_1 + \cdots + w_d
\end{aligned}$$

[122] Image from Hannah Markwig.

Vice versa, if

$$\begin{aligned}
&p \in -(-v+\mathit{\Pi}) + w_1 + \cdots + w_d \\
\Rightarrow &p - w_1 - \ldots - w_d \in -(-v+\mathit{\Pi}) \\
\Rightarrow &-p + w_1 + \ldots + w_d \in -v + \mathit{\Pi} \\
\Rightarrow &-p + w_1 + \ldots + w_d = -v + \lambda_1 w_1 + \ldots + \lambda_d w_d
\end{aligned}$$

for some $0 < \lambda_i < 1$, so

$$\begin{aligned}
\Rightarrow &p - w_1 - \ldots - w_d = v - \lambda_1 w_1 - \ldots - \lambda_d w_d \\
\Rightarrow &p = v + (1-\lambda_1) w_1 + \ldots + (1-\lambda_d) w_d
\end{aligned}$$

and $0 < 1 - \lambda_i < 1, \Rightarrow \quad p \in v + \mathit{\Pi}$. It follows that

$$f_{v+\mathit{\Pi}}(z) = f_{-(-v+\mathit{\Pi})}(z) \cdot z^{w_1} \cdot \ldots \cdot z^{w_d} = f_{-v+\mathit{\Pi}}\left(\frac{1}{z_1}, \ldots, \frac{1}{z_d}\right) \cdot z^{w_1} \cdot \ldots \cdot z^{w_d},$$

where the last equality follows since for any set $S \subset \mathbb{R}^d$ (with $-S = \{-x \mid x \in S\}$) we have

$$f_{-S}(z_1, \ldots, z_d) = f_S\left(\frac{1}{z_1}, \ldots, \frac{1}{z_d}\right),$$

because a lattice point $m \in S$ produces a monomial $z_1^{m_1} \cdot \ldots \cdot z_d^{m_d}$ in f_S, which, evaluated at $\frac{1}{z_1}, \ldots, \frac{1}{z_d}$ yields

$$\left(\frac{1}{z_1}\right)^{m_1} \cdot \ldots \cdot \left(\frac{1}{z_d}\right)^{m_d} = z_1^{-m_1} \cdot \ldots \cdot z_d^{-m_d} = z^{-m},$$

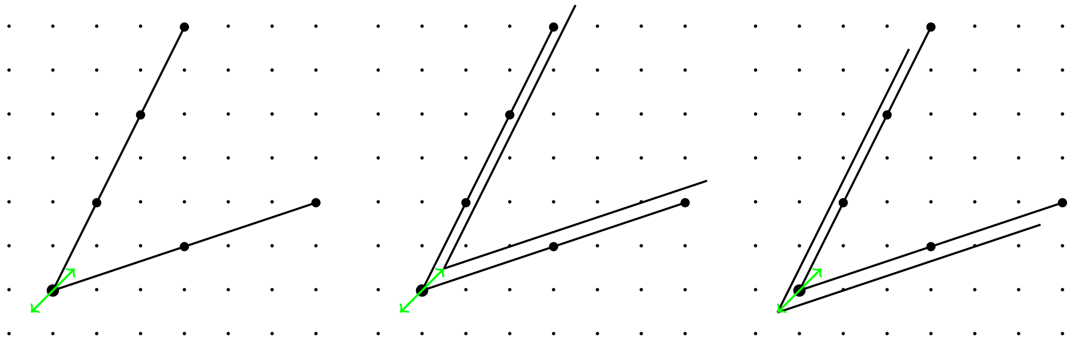
which is the monomial for $-m \in -S$.
We insert $\frac{1}{z}$ in the equation above and obtain

$$f_{v+\mathit{\Pi}}\left(\frac{1}{z}\right) = f_{-v+\mathit{\Pi}}(z) \cdot z^{-w_1} \cdot \ldots \cdot z^{-w_d}.$$

$$\begin{aligned}
\Rightarrow f_{v+\sigma}\left(\frac{1}{z}\right) &= \frac{f_{v+\mathit{\Pi}}\left(\frac{1}{z}\right)}{(1-z^{-w_1}) \cdot \ldots \cdot (1-z^{-w_d})} \\
&= \frac{f_{-v+\mathit{\Pi}}(z) \cdot z^{-w_1} \cdot \ldots \cdot z^{-w_d}}{(1-z^{-w_1}) \cdot \ldots \cdot (1-z^{-w_d})} \\
&= \frac{f_{-v+\mathit{\Pi}}(z)}{z^{w_1}(1-z^{-w_1}) \cdot \ldots \cdot z^{w_d}(1-z^{-w_d})} \\
&= \frac{f_{-v+\mathit{\Pi}}(z)}{(z^{w_1}-1) \cdot \ldots \cdot (z^{w_d}-1)} \\
&= (-1)^d \frac{f_{-v+\mathit{\Pi}}(z)}{(1-z^{w_1}) \cdot \ldots \cdot (1-z^{w_d})} \\
&= (-1)^d f_{-v+\sigma}(z).
\end{aligned}$$

7.58 Proposition. *Let σ be any d-dim strictly convex rational cone. Then*

$$f_\sigma\left(\frac{1}{z}\right) = (-1)^d f_{\sigma^\circ}(z).$$



(a) σ (b) $v+\sigma$ (c) $-v+\sigma$

Abbildung 119: $v+\sigma$ contains all interior lattice points and $-v+\sigma$ contains all lattice points

Beweis. We triangulate σ into $\sigma_1, \dots, \sigma_m$. Similarly to before, we can find $v \in \mathbb{R}^n$ s.th. the shifted cone $v+\sigma$ contains the interior lattice points of σ,

$$\sigma^\circ \cap \mathbb{Z}^d = (v+\sigma) \cap \mathbb{Z}^d,$$

and s.th. no lattice point lies on the boundary of any of the shifted simplicial cones $v+\sigma_i$ or $-v+\sigma_i$. It follows that

$$\sigma \cap \mathbb{Z}^d = (-v+\sigma) \cap \mathbb{Z}^d.$$

Then, using the proposition 7.57:

$$\begin{aligned} f_\sigma\left(\frac{1}{z}\right) &= f_{-v+\sigma}\left(\frac{1}{z}\right) \\ &= \sum_{j=1}^{m} f_{-v+\sigma_j}\left(\frac{1}{z}\right) = \sum_{j=1}^{m}(-1)^d f_{v+\sigma_j}(z) \\ &= (-1)^d f_{v+\sigma}(z) = (-1)^d f_{\sigma^\circ}(z) \end{aligned}$$

□

7.59 Theorem. *Let P be a lattice polytope. Then*

$$\mathrm{Ehr}_P\left(\frac{1}{z}\right) = (-1)^{\dim P+1}\mathrm{Ehr}_{P^\circ}(z) = \sum_{t\geq 1} L_{P^\circ}(t) z^t.$$

Beweis. Let $\dim P = d$. We know that

$$\mathrm{Ehr}_P(z) = \sum_{t\geq 0} L_P(t) z^t = f_{\mathrm{Cone}(P)}(1, \dots, 1, z).$$

Analogously,

$$\mathrm{Ehr}_{P^\circ}(z) = f_{\mathrm{Cone}(P)^\circ}(1, \dots, 1, z).$$

Proposition 7.58 implies

$$f_{\mathrm{Cone}(P)^\circ}(1, \dots, 1, z) = (-1)^{d+1} \cdot f_{\mathrm{Cone}(P)}(1, \dots, 1, \frac{1}{z}).$$

The claim follows.

$$\mathrm{Ehr}_P\left(\frac{1}{z}\right) = f_\sigma(1, \dots, 1, \frac{1}{z}) = (-1)^{d+1} f_{\sigma^\circ}(1, \dots, 1, z) = (-1)^{d+1}\mathrm{Ehr}_{P^\circ}(z)$$

□

7.60 Lemma. *Let $Q(t)$ be a polynomial. Let*

$$R_Q^+(z) = \sum_{t\geq 0} Q(t)z^t,$$
$$R_Q^-(z) = \sum_{t<0} Q(t)z^t.$$

Then $R_Q^+(z) + R_Q^-(z) = 0$ as rational functions.

Beweis. Assume first $Q(t) = 1$. Then

$$\begin{aligned}
R_Q^+(z) &= \sum_{t\geq 0} z^t = \frac{1}{1-z}, \\
R_Q^-(z) &= \sum_{t<0} z^t = \sum_{t>0}\left(\frac{1}{z}\right)^t = \frac{1}{1-\frac{1}{z}} - 1 \quad \text{and} \\
R_Q^+(z) + R_Q^-(z) &= \frac{1}{1-z} + \frac{1}{1-\frac{1}{z}} - 1 = \frac{1}{1-z} + \frac{z}{z-1} - 1 \\
&= \frac{1}{1-z} - \frac{z}{1-z} - 1 = \frac{1-z}{1-z} - 1 = 0.
\end{aligned}$$

Next, consider a monomial $Q(t) = t^d$.

$$\begin{aligned}
R_Q^+(z) &= \sum_{t\geq 0} t^d z^t = \frac{\sum_{k=1}^d A(d,k)z^k}{(1-z)^{d+1}} \\
R_Q^-(z) &= \sum_{t<0} t^d z^t = \sum_{t>0} (-t)^d \left(\frac{1}{z}\right)^t \\
&= (-1)^d \sum_{t>0} t^d \left(\frac{1}{z}\right)^t = (-1)^d \frac{\sum_{k=1}^d A(d,k)\left(\frac{1}{z}\right)^k}{(1-\frac{1}{z})^{d+1}} \\
R_Q^+(z) + R_Q^-(z) &= \frac{\sum_{k=1}^d A(d,k)z^k}{(1-z)^{d+1}} + (-1)^d \frac{z^{d+1}\sum_{k=1}^d A(d,k)\left(\frac{1}{z}\right)^k}{z^{d+1}(1-\frac{1}{z})^{d+1}} \\
&= \frac{\sum_{k=1}^d A(d,k)z^k}{(1-z)^{d+1}} + (-1)^d \frac{\sum_{k=1}^d A(d,k)z^{d+1-k}}{(z-1)^{d+1}} \\
&= \frac{\sum_{k=1}^d A(d,k)z^k - \sum_{k=1}^d A(d,k)z^{d+1-k}}{(1-z)^{d+1}} \\
&= \frac{\sum_{k=1}^d A(d,k)z^k - \sum_{k=1}^d A(d,d+1-k)z^k}{(1-z)^{d+1}} \\
&= \frac{\sum_{k=1}^d (A(d,k) - A(d,d+1-k))z^k}{(1-z)^{d+1}} = 0.
\end{aligned}$$

as $A(d,k) = A(d,d+1-k)$.
The claim for a general polynomial $Q(t)$ follows by adding the contributions for the terms. □

Now we can prove the reciprocity theorem 7.55:

Beweis.

$$
\begin{aligned}
\mathrm{Ehr}_P\left(\frac{1}{z}\right) &= \sum_{t\geq 0} L_P(t)\left(\frac{1}{z}\right)^t \\
&= \sum_{t\geq 0} L_P(t)z^{-t} = \sum_{t\leq 0} L_P(-t)z^t \\
&= 1 + \sum_{t<0} L_P(-t)z^t \\
&\overset{\text{use lemma 7.60}}{\underset{\text{with } Q(t)=L_P(-t)}{=}} 1 - \sum_{t\geq 1} L_P(-t)z^t = -\sum_{t\geq 1} L_P(-t)z^t.
\end{aligned}
$$

Using the reciprocity theorem for Ehrhart series we just proved, we have

$$
\begin{aligned}
\mathrm{Ehr}_{P^\circ}(z) &= \sum_{t\geq 1} L_{P^\circ}(t)z^t \\
&= (-1)^{d+1}\mathrm{Ehr}_P\left(\frac{1}{z}\right) \\
&\overset{\text{by the above}}{=} (-1)^d \sum_{t\geq 1} L_P(-t)z^t.
\end{aligned}
$$

We compare the coefficient of z^t and obtain:

$$
L_{P^\circ}(t) = (-1)^d L_P(-t).
$$

□

We can apply reciprocity to deduce:

7.61 Theorem. *Let P be a d-dim lattice polytope with*

$$
Ehr_P(z) = \frac{h_d z^d + \ldots + h_1 z + h_0}{(1-z)^{d+1}}.
$$

Then $h_d = h_{d-1} = \ldots = h_{k+1} = 0$ and $h_k \neq 0 \Leftrightarrow (d-k+1)P$ is the smallest integer stretch of P which contains an interior lattice point.

Beweis. We know

$$
\begin{aligned}
&h_d = \ldots = h_{k+1} = 0 \text{ and } h_k \neq 0 \\
\Leftrightarrow\ & L_P(-1) = L_P(-2) = \ldots = L_P(-(d-k)) = 0 \text{ and } L_P(-(d-k+1)) \neq 0 \\
\Leftrightarrow\ & L_{P^\circ}(1) = \ldots = L_{P^\circ}(d-k) = 0 \text{ and } L_{P^\circ}(d-k+1) \neq 0 \\
\Leftrightarrow\ & (d-k+1)P \text{ is the smallest integer stretch of } P \\
& \text{which contains an interior lattice point.}
\end{aligned}
$$

□

7.62 Theorem (Hibi's palindrome theorem)**.** *Let P be a d-dim lattice polytope s.th. $0 \in P^\circ$ and with*

$$\mathrm{Ehr}_P(z) = \frac{h_d z^d + \ldots + h_1 z + h_0}{(1-z)^{d+1}}.$$

Then P is reflexive if and only if

$$h_k = h_{d-k} \quad \forall \quad 0 \le k \le \frac{d}{2}.$$

To prepare for the proof, we need:

7.63 Lemma. *Let $a_1, \ldots, a_d, b \in \mathbb{Z}$ with*

$$\gcd(a_1, \ldots, a_d, b) = 1 \quad \textit{and} \quad b > 1.$$

Then $\exists\ c, t \in \mathbb{Z}_{>0}$ such that

$$tb < c < (t+1)b$$

and

$$\{(m_1, \ldots, m_d) \in \mathbb{Z}^d \mid a_1 m_1 + \cdots + a_d m_d = c\} \neq \emptyset.$$

Beweis. Let $g = \gcd(a_1, \ldots, a_d)$.
We have $\gcd(g, b) = 1 \Rightarrow \exists\ k, t \in \mathbb{Z}$ such that $kg - tb = 1$.
By possibly adding multiples of bg, i.e., using $bg - g \cdot b = 0$, we can assume that in the equation above, $t > 0$.
Let $c = kg$. Then $tb = kg - 1 = c - 1 < c$.
Since $b > 1$, $\quad tb + b > tb + 1 = c \Rightarrow tb < c < (t+1)b$. Since

$$g = \gcd(a_1, \ldots, a_d), \quad \exists\, m_1, \ldots, m_d \in \mathbb{Z} : a_1 m_1 + \cdots + a_d m_d = kg = c.$$

□

We know: P is reflexive if and only if

$$P = \{u \mid Au \le 1\}$$

for an integer matrix A.

7.64 Lemma. *Let P be a lattice polytope, $0 \in P^\circ$,*

$$P = \{u \mid Au \le 1\}$$

for an integer matrix A if and only if

$$P^\circ \cap \mathbb{Z}^d = \{0\}$$

and for all $t \in \mathbb{Z}_{>0}$,

$$(t+1)P^\circ \cap \mathbb{Z}^d = tP \cap \mathbb{Z}^d.$$

This condition means that the only lattice points we gain when passing from tP to $(t+1)P$ are the ones on the boundary of $(t+1)P$.

Beweis. "$\Rightarrow$ "

$$\begin{aligned}
P^\circ &= \{x \mid Ax < 1\}\\
\mathcal{E}P &= \{\mathcal{E}x \mid Ax \le 1\} = \{\mathcal{E}x \mid A(\mathcal{E}x) \le \mathcal{E}\} = \{x' \mid Ax' \le \mathcal{E}\}\\
\lim_{\mathcal{E}\to 0} \mathcal{E}P &= \{x' \mid Ax' \le \mathcal{E}\} = \{0\}
\end{aligned}$$

An integer point in P° satisfies $Ax \le 0 < 1 \Rightarrow$ it is 0.
Furthermore,

$$\begin{aligned}
(t+1)P^\circ &= \{x \mid Ax < t+1\},\\
tP &= \{x \mid Ax \le t\}.
\end{aligned}$$

Any lattice point in $(t+1)P^\circ$ is already contained in tP
"$\Leftarrow$ " Let $P^\circ \cap \mathbb{Z}^d = \{0\}$ and for all $t \in \mathbb{Z}_{>0}$,

$$(t+1)P^\circ \cap \mathbb{Z}^d = tP \cap \mathbb{Z}^d.$$

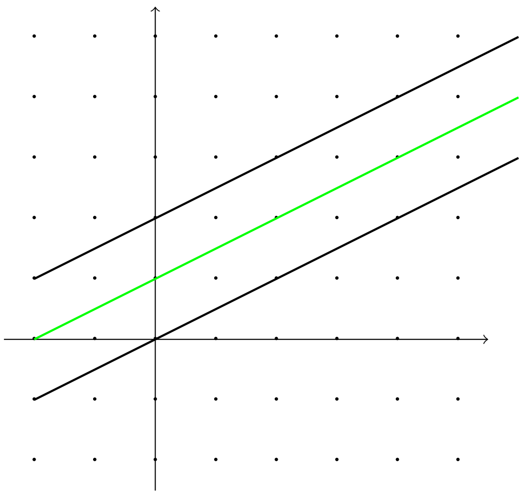
Let H be a hyperplane defining a facet F of P. Assume

$$H = \{x \in \mathbb{R}^d \mid \langle a, x\rangle = b\}$$

for $a \in \mathbb{Z}^d$ such that $\gcd(a_1, \dots, a_d) = 1$. The points in P satisfy $\langle a, x\rangle \le b$.
The points in tP are $t \cdot x$ for x satisfying $\langle a, x\rangle \le b$

$$\begin{aligned}
&\Rightarrow \langle a, tx\rangle \le tb\\
&\Rightarrow \text{points } x' \text{ in } tP \text{ satisfy } \langle a, x'\rangle \le tb\\
&\Rightarrow \text{the hyperplane } tH \text{ is a defining hyperplane for } tP.
\end{aligned}$$

We can pick t large enough such that tF (which is the facet of tP corresponding to F) contains interior lattice points.
If there were lattice points between tH and $(t+1)H$, we could fit a lattice shift L of $H \cap \mathbb{Z}^d$ between tH and $(t+1)H$.



Since tF already contained interior lattice points, $L \cap (t+1)P^\circ$ also contains lattice points

$$\Rightarrow (t+1)P^\circ \cap \mathbb{Z}^d \setminus tP \cap \mathbb{Z}^d \neq \emptyset. \lightning$$

Thus, there are no lattice points between tH and $(t+1)H$.

$$\Rightarrow \{x \in \mathbb{Z}^d \mid tb < \langle a, x\rangle < (t+1)b\} = \emptyset.$$

If $b > 1$, lemma 7.64 yields a contradiction $\Rightarrow b = 1 \Rightarrow$ the defining hyperplane takes the form $\langle a, x\rangle = 1$. Since this holds for all hyperplanes, we deduce that P has the form

$$P = \{u \mid Au \leq 1\}$$

for an integer matrix A. $\square$ $\square$

We are ready to prove Hibi's palindrome theorem 7.62:

Beweis.

$$P \text{ is reflexive } \Leftrightarrow P^\circ \cap \mathbb{Z}^d = \{0\}$$

and for all $t \in \mathbb{Z}_{>0}$,

$$(t+1)P^\circ \cap \mathbb{Z}^d = tP \cap \mathbb{Z}^d$$

$$\begin{aligned}
\Leftrightarrow \sum_{t\geq 1} L_P(t-1)z^t &= \sum_{t\geq 0} L_{P^\circ}(t)z^t \\
&= \mathrm{Ehr}_{P^\circ}(z) = (-1)^{d+1}\mathrm{Ehr}_P\left(\frac{1}{z}\right) \\
&= (-1)^{d+1}\frac{h_d\left(\frac{1}{z}\right)^d + \ldots + h_1\left(\frac{1}{z}\right) + h_0}{(1-\frac{1}{z})^{d+1}} \\
&= (-1)^{d+1}\cdot\frac{z^{d+1}\left(h_d\left(\frac{1}{z}\right)^d + \ldots + h_1\left(\frac{1}{z}\right) + h_0\right)}{z^{d+1}(1-\frac{1}{z})^{d+1}} \\
&= (-1)^{d+1}\cdot\frac{h_d z + \ldots + h_1 z^d + h_0 z^{d+1}}{(z-1)^{d+1}}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\sum_{t\geq 1} L_P(t-1)z^t &= z\sum_{t\geq 0} L_P(t)z^t = z\mathrm{Ehr}_P(z) \\
&= \frac{h_d z^{d+1} + \cdots + h_0 z}{(1-z)^{d+1}} = \frac{h_0 z + \cdots + h_d z^{d+1}}{(1-z)^{d+1}}.
\end{aligned}$$

$$\begin{aligned}
&\Leftrightarrow h_d z + \ldots + h_0 z^{d+1} = h_0 z + \ldots + h_d z^{d+1} \\
&\Leftrightarrow h_k = h_{d-k} \quad \text{for all} \quad 0 \leq k \leq \frac{d}{2}.
\end{aligned}$$

$\square$

7.65 Example. Remember the Ehrhart series of a d-dim diamond is

$$\mathrm{Ehr}_P(z) = \frac{(1+z)^d}{(1-z)^{d+1}} = \frac{\sum_{k=0}^{d} \binom{d}{k} z^k}{(1-z)^{d+1}}.$$

Since

$$\binom{d}{k} = \binom{d}{d-k},$$

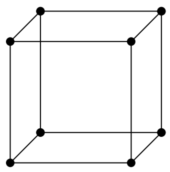
we conclude that any diamond is reflexive. We saw that before for $d = 2$.

7.66 Definition. Let P be a polytope. We denote by $f_k := \#k - \text{dim}$ faces of P, called the **face numbers** for P.

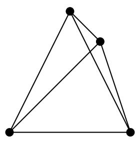
7.67 Theorem. *Let P be a d-dim lattice polytope. Then*

$$\sum_{j=0}^{d} (-1)^j f_j = 1.$$

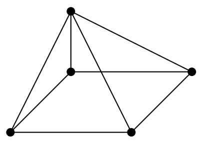
7.68 Example. .



vertices facets
$8 \;-\; 12 \;+\; 6 \;-\; 1 = 1$
edges P



$4 - 6 + 4 - 1 = 1$



$5 - 8 + 5 - 1 = 1$

Beweis. We count lattice points in tP, organizing them in terms of the faces in which relative interior they are contained. The relative interior of a vertex is the vertex.

$$\begin{aligned} L_P(t) &= \sum_{\substack{F \neq \emptyset \\ F \text{ face of } P}} L_{F^\circ}(t) \\ &\overset{\text{reciprocity}}{=} \sum_{\substack{F \neq \emptyset \\ F \text{ face of } P}} (-1)^{\dim F} L_F(-t). \end{aligned}$$

Inserting 0, we obtain

$$L_P(0) = \sum_{\substack{F \neq \emptyset \\ F \text{ face of } P}} (-1)^{\dim F} \cdot 1 = \sum_{j=0}^{d} (-1)^j \cdot f_j.$$

□

We end the chapter with an application of Ehrhart theory to magical squares.

7.69 Definition. A **magical square** is an $n \times n$-matrix with nonnegative integer entries s.th. the sums of all entries in a row, column, or diagonal coincide. We call it **semimagical** if the sums of the rows and columns coincide.

7.70 Example. .

$$\begin{array}{ccccc}
 & \boxed{4} & \boxed{9} & \boxed{2} & \longrightarrow 15 \\
 & \boxed{3} & \boxed{5} & \boxed{7} & \longrightarrow 15 \\
 & \boxed{8} & \boxed{1} & \boxed{6} & \longrightarrow 15 \\
\swarrow & \downarrow & \downarrow & \downarrow & \searrow \\
15 & 15 & 15 & 15 & 15
\end{array}
\qquad
\begin{array}{ccccc}
 & \boxed{1} & \boxed{2} & \boxed{0} & \longrightarrow 3 \\
 & \boxed{0} & \boxed{1} & \boxed{2} & \longrightarrow 3 \\
 & \boxed{2} & \boxed{0} & \boxed{1} & \longrightarrow 3 \\
\swarrow & \downarrow & \downarrow & \downarrow & \searrow \\
3 & 3 & 3 & 3 & 3
\end{array}
\qquad
\begin{array}{cccc}
\boxed{3} & \boxed{0} & \boxed{0} & \longrightarrow 3 \\
\boxed{0} & \boxed{1} & \boxed{2} & \longrightarrow 3 \\
\boxed{0} & \boxed{2} & \boxed{1} & \longrightarrow 3 \\
\downarrow & \downarrow & \downarrow & \\
3 & 3 & 3 &
\end{array}$$

The first matrix is **magical** (and particularly nice, since it contains all numbers $1, \dots, 9$ precisely once). The second is also **magical**, while the third is **semimagical**.

<u>Problem:</u> Count (semi-)magical squares. (If we restricted to consider only those with each entry $1, \dots, n^2$ precisely once (following the tradition). This is known only for $n \leq 5$.

7.71 Example. Let $n = 2$, denote the magical sum by t.

$$\begin{array}{|c|c|}\hline s & t-s \\ \hline t-s & s \\ \hline\end{array} \quad \begin{array}{|c|c|}\hline \frac{t}{2} & \frac{t}{2} \\ \hline \frac{t}{2} & \frac{t}{2} \\ \hline\end{array}$$

The left one is semimagical for any $0 \leq s \leq t$, the right one is only magical (for t even) (2s=t)

$$H_2(t) := \# \text{ semimagical squares of size 2 with sum } t = t + 1.$$

$$M_2(t) := \# \text{ magical squares of size 2 with sum } t = \begin{cases} 1, & t \text{ even} \\ 0, & \text{else.} \end{cases}$$

7.72 Definition.

$$B_n = \left\{ \begin{pmatrix} x_{11} & \dots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{n1} & \dots & x_{nn} \end{pmatrix} \in \mathbb{R}^{n \times n} \;\middle|\; \begin{array}{ll} x_{ij} \geq 0 & \\ \sum_j x_{jk} = 1 & \text{for all } 1 \leq k \leq n \\ \sum_k x_{jk} = 1 & \text{for all } 1 \leq j \leq n \end{array} \right\}$$

is the n-th **Birkhoff-von Neumann polytope**.

7.73 Example. $B_2 \subset \mathbb{R}^4$ is the segment

$$\operatorname{Conv}\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right),$$

it contains all

$$\begin{pmatrix} t & 1-t \\ 1-t & t \end{pmatrix} \quad \text{for } 0 \leq t \leq 1.$$

Fact: The vertices of B_n are the permutation matrices.
It follows that B_n is a lattice polytope.
If we view the points in B_n as column vectors in $\mathbb{R}^{n^2}$ rather than as matrices, we can write

$$B_n = \{x \in \mathbb{R}^{n^2}_{\geq 0} \mid Ax = b\},$$

for

$$A = \begin{pmatrix}
1 & 1 & \dots & 1 & & & & & & & & & & \\
 & & & & 1 & \dots & 1 & & & & & & & \\
 & & & & & & & 1 & \dots & 1 & & & & \\
 & & & & & & & & & & \ddots & & & \\
 & & & & & & & & & & & 1 & \dots & 1 \\
1 & & & & 1 & & & 1 & \dots & & & 1 & & \\
 & 1 & & & & & & & & & & & & \\
 & & \ddots & & & \ddots & & & \ddots & & & & \ddots & \\
 & & & 1 & & & 1 & & & 1 & & & & 1
\end{pmatrix} \in \mathbb{Z}^{2n \times n^2}$$

and $b = (1, \dots, 1)$.

$$\begin{pmatrix}
1 & -1 & 0 & \dots & 0 \\
-1 & 1 & 0 & \dots & 0 \\
0 & 0 & & \dots & 0 \\
\vdots & \vdots & & & 0 \\
0 & 0 & & \dots & 0
\end{pmatrix}$$

(viewed as column vector) is in the kernel of A, also:

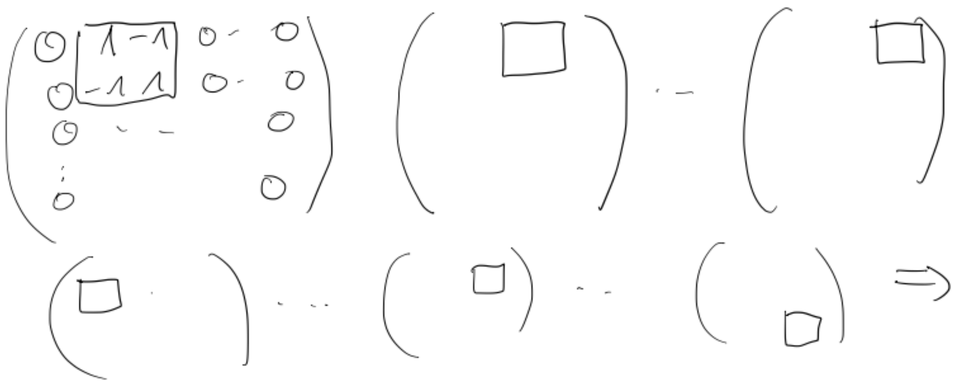


Abbildung 120: Matrices in the kernel of A [123]

There are $(n-1)^2$ linearly independent vectors in $\ker(A)$.

$$\begin{aligned}
&\operatorname{rk}(A) + \dim\ker(A) = n^2 \\
\Rightarrow &\operatorname{rk}(A) \leq n^2 - (n-1)^2 = n^2 - n^2 + 2n - 1 = 2n - 1.
\end{aligned}$$

By subtracting the last n rows of the first, we obtain:

[123] Image from Hannah Markwig.

$$\begin{pmatrix}
 & & & & -1 & \dots & -1 & -1 & \dots & -1 & \dots & -1 & \dots & -1 \\
 & & & & 1 & \dots & 1 & & & & & & & \\
 & & & & & & & 1 & \dots & 1 & & & & \\
 & & & & & & & & & & \ddots & & & \\
 & & & & & & & & & & & 1 & \dots & 1 \\
1 & & & & 1 & & & 1 & \dots & & & 1 & & \\
 & 1 & & & & & & & & & & & & \\
 & & \ddots & & & \ddots & & & \ddots & & & & \ddots & \\
 & & & 1 & & & 1 & & & 1 & & & & 1
\end{pmatrix}$$

Furthermore, subtracting rows number $2, \dots, n$, we obtain 0 0-row. Reordering the rows, the first ones in each row are pivots.

$$\begin{aligned}
&\Rightarrow \mathrm{rk}(A) = 2n - 1 \\
&\Rightarrow \dim(B_n) = n^2 - 2n + 1 = (n-1)^2
\end{aligned}$$

7.74 Theorem. *The number of semimagical squares of size n with magical sum t is a polynomial in t of degree $(n-1)^2$.*

Beweis.

$$\begin{aligned}
H_n(t) &= \#\text{ semimagical squares of size } n \text{ with magical sum } t \\
&= \#(tB_n \cap \mathbb{Z}^{n^2}) = L_{B_n}(t), \\
\text{and } \dim(B_n) &= (n-1)^2.
\end{aligned}$$

□

7.75 Lemma.

$$\begin{aligned}
H_n^\circ(t) :=& \#\textit{ semimagical squares of size } n \textit{ with magical sum } t \textit{ with positive entries} \\
=& H_n(t-n).
\end{aligned}$$

Beweis. Given a semimagical square with positive entries, we can subtract 1 from each entry and obtain a semimagical square (with not necessarily positive, but nonnegative entries) and with magical sum $n - t$.
Vice versa, from any semimagical square with nonnegative entries and sum $n - t$, we can produce one with positive entries and sum t by adding 1 to each entry. This is a bijection between the two sets we are counting. □

7.76 Lemma.

$$H_n^\circ(1) = \ldots = H_n^\circ(n-1) = 0.$$

Beweis. This follows, since we have at least sum n if we have only positive entries. □

7.77 Theorem.

$$H_n(-n-t) = (-1)^{(n-1)^2} H_n(t)$$

and

$$H_n(-1) = H_n(-2) = \ldots = H_n(-n+1) = 0.$$

Beweis.

$$\begin{aligned}
&H_n^\circ(t) = L_{B_n^\circ}(t) \overset{\text{reciprocity}}{=} (-1)^{(n-1)^2} L_{B_n}(-t) = (-1)^{(n-1)^2} H_n(-t) \\
\Rightarrow &H_n^\circ(-t) = (-1)^{(n-1)^2} H_n(t).
\end{aligned}$$

Since $H_n^\circ(-t) = H_n(-t-n)$ by a previous lemma, the first statement follows. Since

$$H_n^\circ(1) = \ldots = H_n^\circ(n-1) = 0,$$

we conclude using reciprocity that

$$H_n(-1) = H_n(-2) = \ldots = H_n(-n+1) = 0.$$

□

7.78 Corollary. *The Ehrhart series of B_n has the form*

$$\mathrm{Ehr}_{B_n}(z) = \frac{h_{(n-1)(n-2)} z^{(n-1)(n-2)} + \cdots + h_0}{(1-z)^{(n-1)^2+1}}.$$

with the palindromic behaviour

$$h_k = h_{(n-1)(n-2)-k} \quad \text{for } 0 \leq k \leq \frac{(n-1)(n-2)}{2}.$$

Beweis. The fact that

$$h_{(n-1)^2} = \ldots = h_{(n-1)^2-(n-2)} = 0$$

follows since

$$H_n(-1) = \ldots = H_n(-n+1) = 0.$$

The *palindromic behaviour* follows since

$$\begin{aligned}
\mathrm{Ehr}_P\left(\frac{1}{z}\right) =& (-1)^{d+1}\mathrm{Ehr}_{P^\circ}(z) \\
=& (-1)^{d+1} \sum_{t \geq k} L_{P^\circ}(t) z^t \\
=& (-1)^{d+1} \sum_{t \geq k} L_P(t-k) z^t \\
\overset{t'=t-k}{=}& (-1)^{d+1} \sum_{t' \geq 0} L_P(t') z^{t'+k} \\
=& (-1)^{d+1} \cdot z^k \cdot \mathrm{Ehr}_P(z).
\end{aligned}$$

□

7.79 Lemma.

$$H_n(1) = n!.$$

Beweis.

$$H_n(1) = \#\text{ vertices of } B_n = \#\text{ permutation matrices}.$$

□

7.80 Example. We can use these results to find $H_n(t)$ by *interpolation*.
We do that for $n = 3$. We know

$$\begin{aligned} H_3(0) &= 1, \\ H_3(-1) &= H_3(-2) = 0 \\ H_3(-3) &= H_3(-3-0) = (-1)^4 \cdot H_3(0) = 1 \\ H_3(1) &= 3! = 6. \end{aligned}$$

Furthermore, $H_3(t)$ is a polynomial of degree 4. Let

$$H_3(t) = c_4 t^4 + \cdots + c_0 \Rightarrow H_3(0) = c_0 = 1,$$

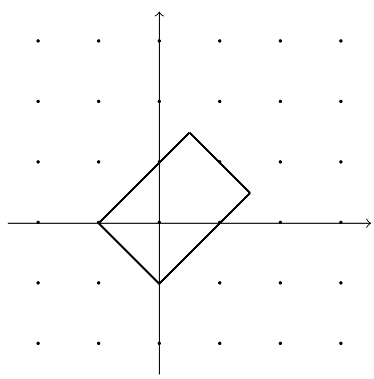
$$\begin{aligned} H_3(-1) &= c_4 - c_3 + c_2 - c_1 + 1 = 0 \\ H_3(-2) &= 16c_4 - 8c_3 + 4c_2 - 2c_1 + 1 = 0 \\ H_3(-3) &= 81c_4 - 27c_3 + 9c_2 - 3c_1 + 1 = 0 \\ H_3(1) &= c_4 + c_3 + \cdots + c_0 = 6. \end{aligned}$$

We can solve this 5×5 linear system of equations and obtain

$$H_3(t) = \frac{1}{8}t^4 + \frac{3}{4}t^3 + \frac{15}{8}t^2 + \frac{9}{4}t + 1.$$

<u>Outlook:</u> Magical squares, Ehrhart theory for rational polytopes.
If we intersect B_n with the two additional hyperplanes for the diagonals, the polytope will not be a lattice polytope anymore.
Being defined by integer hyperplanes, it will still be a rational polytope.

7.81 Example. $\{y + x \leq 2 \quad -y - x \leq 1 \quad y - x \leq 1 \quad -y + x \leq 1\}$ is defined by integer hyperplanes, but is not a lattice polytope.



Its vertices $(-1, 0), (0, -1), (\frac{1}{2}, \frac{3}{2}), (\frac{3}{2}, \frac{1}{2})$ are in $\mathbb{Q}^2$, but not in $\mathbb{Z}^2$.

There is an Ehrhart theory for rational polytopes as well.
For such polytopes, $L_P(t)$ is not polynomial in t, but a **quasipolynomial**, i.e., it has periodic behavior which is polynomial, as we saw in the example of $M_2(t)$.

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