Exercise sheet 1

Nonlinear Dispersive PDEs
Sommersemester 2018
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Exercise 1 (5pt). HLS Inequality

Prove the Hardy-Littlewood-Sobolev inequality (Corollary i.5 of the lecture):

$$\forall 0 < \alpha < d,\ \forall (p,r) \in]1,\infty[^2 \text{ such that } \boxed{\frac{1}{p} + \frac{\alpha}{d} = 1 + \frac{1}{r}}:$$

$$\exists C > 0\ ,\ \forall f \in L^p(\mathbb{R}^d)\ ,\ \| |\cdot|^{-\alpha} * f \|_r \leq C \| f \|_p\ .$$

You may use the refined Young's inequality.

Exercise 2 (4pt). Weakly integrable but not integrable functions

Prove that for any $1 \leq p < \infty$, $L^p(\mathbb{R}^d) \subset L^p_{\mathrm{w}}(\mathbb{R}^d)$ is a *strict* inclusion. (Hint: think of $|\cdot|^{\lambda}$)

Exercise 3 (6pt). The weak L^p norm is a quasi-norm

Verify that the weak L^p norm, defined as

$$\| g \|_{p,\mathrm{w}}^p := \sup_{\lambda>0} \lambda^p \int_{\{|g(x)|>\lambda\}} \mathrm{d}x$$

satisfies the properties of a quasi-norm (the triangle inequality is true up to a constant).

Exercise 4 (7pt). Properties of the Fourier transform

Prove that for any $f, g \in \mathscr{S}(\mathbb{R}^d)$, and any $\alpha \in \mathbb{N}^d$:

- $\left(\frac{i\partial}{2\pi}\right)^{\alpha} \hat{f} = \widehat{\left(x^{\alpha} f\right)}$;
- $(ix)^{\alpha} \hat{f} = \widehat{\left(\left(\frac{\partial}{2\pi}\right)^{\alpha} f\right)}$;
- $\widehat{(f * g)} = \hat{f} \cdot \hat{g}$;
- $\widehat{(f \cdot g)} = \hat{f} * \hat{g}$.

Exercise 5 (3pt). Gaussian Mollifier

Prove that $\left(\frac{1}{\pi\varepsilon}\right)^{\frac{d}{2}} e^{-\frac{1}{\varepsilon}|\cdot|^2}$ is a mollifier, *i.e.* that for any measurable function f such that $\exists N \in \mathbb{N}$, $(1 + |\cdot|)^{-N} f(\cdot) \in L^1(\mathbb{R}^d)$:

$$\left(\tfrac{1}{\pi\varepsilon}\right)^{\frac{d}{2}} \left(e^{-\frac{1}{\varepsilon}|\cdot|^2} * f\right)(x) \underset{\varepsilon\to 0}{\longrightarrow} f(x) \qquad \text{(for almost every } x \in \mathbb{R}^d) \ .$$

Exercise 6 (5pt). Compactly supported distributions

Prove that any $T \in \mathscr{D}'(\mathbb{R}^d)$ with $\mathrm{supp}(T)$ compact is tempered (*i.e.* $T \in \mathscr{S}'(\mathbb{R}^d)$).