Exercise sheet 3

Nonlinear Dispersive PDEs
Sommersemester 2018
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Exercise 1 (8pt). Inequalities II

[Justify your answers]

- Let $v \in L^1(\mathbb{R}^5)$, $w \in \dot{H}^4_1(\mathbb{R}^5)$. For which $1 \leq p \leq \infty$ is it true that for any $u \in L^p(\mathbb{R}^5)$, $u * v * w \in L^{10}(\mathbb{R}^5)$? [Recall that Sobolev's embedding $\dot{H}^\sigma_r(\mathbb{R}^d) \hookrightarrow L^{\frac{rd}{d-r\sigma}}(\mathbb{R}^d)$ implies that $\|f\|_{\frac{rd}{d-r\sigma}} \leq C\|f\,;\dot{H}^\sigma_r\|$ for any $f \in \dot{H}^\sigma_r(\mathbb{R}^d)$.]
- Let $u \in H^2(\mathbb{R})$, $v \in H^{-2}(\mathbb{R})$. Does $u * v \in L^\infty(\mathbb{R})$? [*Hints:* $\|u * v\|_\infty \leq \|(\widehat{u * v})\|_1$ (why?), and $1 = \langle\xi\rangle^{-\delta}\langle\xi\rangle^{\delta}$ for any $\delta \geq 0$ and $\xi \in \mathbb{R}$.]

Exercise 2 (7pt). Characteristic functions

Prove that the characteristic function $\chi_{[-\varrho,\varrho]}$ of the interval $[-\varrho,\varrho]$ ($\varrho > 0$) does not belong to $H^\delta(\mathbb{R})$, $\frac{1}{2} \leq \delta \leq 1$, but it belongs to $H^\delta(\mathbb{R})$ for $0 \leq \delta < \frac{1}{2}$.

Exercise 3 (15pt). Contractions

Let $X = H^1(\mathbb{R}^d)$, $\mathscr{X}(I) = C^0(I, X)$. Consider the map $A(t_0, u_0)$, $t_0 \in \mathbb{R}$, $u_0 \in X$, defined as: $\forall u \in \mathscr{X}(I)$

$$[A(t_0,u_0)u](t,x) = e^{i(t-t_0)}u_0(x) - i\int_{t_0}^t e^{i(\tau-t_0)}(V * u(\tau))(x)u(\tau,x)\mathrm{d}\tau \;,$$

where $V \in L^2(\mathbb{R}^d)$. For any $\varrho > 0$, find $T(\varrho) > 0$ such that for any $u_0 \in H^1$: $\|u_0\,;H^1\| \leq \varrho$, then $A(t_0,u_0)$ is a *strict contraction* on $B(I, 2\varrho)$, where $I = [t_0 - T(\varrho), t_0 + T(\varrho)]$.

More precisely, you should prove the following steps (if you are not able to prove one, you may use it to prove the following ones):

- Prove that the gradient acts on $(f * g)$, $f \in L^2$ and $g \in H^1$, as follows: $\nabla(f * g) = f * (\nabla g)$ [*Hint:* use the properties of the Fourier transform]. Use this information to deduce that for any $f \in L^2$ and $g, h \in H^1$ (be careful to a factor two coming from $\|a+b\|^2 \leq 2(\|a\|^2 + \|b\|^2)$):

$$\|(f*g)h\,;H^1\|^2 = \tfrac{1}{4\pi^2}\|\nabla\,(f*g)h\|_2^2 + \|(f*g)h\|_2^2 \leq \tfrac{1}{2\pi^2}\Big(\|(f*(\nabla g))h\|_2^2 + \|(f*g)\nabla h\|_2^2\Big) + \|(f*g)h\|_2^2\;.$$

  The last bound implies

$$\|(f*g)h\,;H^1\| \leq \tfrac{1}{\sqrt{2}\pi}\Big(\|(f*(\nabla g))h\|_2 + \|(f*g)\nabla h\|_2\Big) + \|(f*g)h\|_2\;.$$

- Use the above bound to prove (remember that $a_1b_1 - a_2b_2 = a_1(b_1 - b_2) + a_1b_2 - a_2b_2 = a_1(b_1 - b_2) + (a_1 - a_2)b_2$) that for any $t > t_0$ (for $t < t_0$ being analogous) and for any

$u_1, u_2 \in \mathscr{X}(I)$:

$$\|[A(t_0,u_0)u_1](t) - [A(t_0,u_0)u_2](t)\ ; H^1\| \le \tfrac{1}{\sqrt{2}\pi}\|V\|_2 \int_{t_0}^{t} \Big(\|\nabla(u_1(\tau) - u_2(\tau))\|_2 \\ (\|u_1(\tau)\|_2 + \|u_2(\tau)\|_2) + (\|\nabla u_1(\tau)\|_2 + \|\nabla u_2(\tau)\|_2)\|u_1(\tau) - u_2(\tau)\|_2 \\ + \sqrt{2}\pi(\|u_1(\tau)\|_2 + \|u_2(\tau)\|_2)\|u_1(\tau) - u_2(\tau)\|_2 \Big) \mathrm{d}\tau\ .$$

From this conclude, taking the supremum for $t \in I$ on both sides, that

$$|A(t_0,u_0)u_1 - A(t_0,u_0)u_2|_I \le \tfrac{1}{\sqrt{2}\pi}\|V\|_2(|u_1|_I + |u_2|_I)(2 + \sqrt{2}\pi)T(\varrho)|u_1 - u_2|_I \\ \le \tfrac{4\varrho}{\pi}(\sqrt{2} + \pi)\|V\|_2 T(\varrho)|u_1 - u_2|_I\ .$$

- Choose the time $T(\varrho)$ in the above expression that gives a strict contraction estimate (with contraction constant $\frac{1}{2}$). Then check that from this choice it also follows that $A(t_0,u_0)u$ maps $B(I, 2\varrho)$ into itself (use the fact that $|A(t_0,u_0)u|_I \le |e^{i(\cdot - t_0)}u_0|_I + |A(t_0,u_0)u - A(t_0,u_0)0|_I$).