Problem 1.1 – Connections

Let ∇ be a connection on a manifold M.

a) Show that ∇ is torsion-free if and only if, in every coordinate chart, the Christoffel symbols satisfy $\Gamma^k_{ij} = \Gamma^k_{ji}$.

b) Suppose that M is equipped with a Riemannian metric g. Show that the Christoffel symbols of the Levi-Civita connection are given by

$$\Gamma^k_{ij} = \frac{1}{2} g^{kl}(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}).$$

Problem 1.2 – Curves

Let (M, g) be a smooth Riemannian manifold and let $\gamma : [a, b] \to M$ be a smooth curve.

a) Show that the length $\ell(\gamma)$ is invariant under reparametrization, i.e., if $u : [c, d] \to [a, b]$ is a smooth map with smooth inverse, then $\ell(\gamma) = \ell(\gamma \circ u)$.

b) Show that there is a reparametrization $\tilde{\gamma}$ of γ with unit speed (i.e., $|\dot{\gamma}(t)| = 1$). It will be helpful to consider the arc length function $s : [a, b] \to [0, \ell(\gamma)]$ defined by

$$s(t) = \int_a^t \mathrm{d}u \, |\dot{\gamma}(u)|.$$

c) Show that for $X, Y \in \mathfrak{X}(\gamma)$, $\dfrac{d}{dt} g(X, Y) = g(D_t X, Y) + g(X, D_t Y)$.

d) Show that if $X \in \mathfrak{X}(\gamma)$ is parallel along γ, then $|\dot{X}| = \sqrt{g(X, X)}$ is constant.

Problem 1.3 – Gradient

Let $M \subset \mathbb{R}^n$ be a manifold with metric g. Given $f \in C^\infty(M)$, we define the vector field $\operatorname{grad} f$ by the relation

$$df(X) = g(\operatorname{grad} f, X).$$

Show that in local coordinates $\operatorname{grad} f$ is given by $\operatorname{grad} f = g^{ij} \partial_i f \partial_j$.

Problem 1.4 – Isometries

Let $\phi : (M, g) \to (\tilde{M}, \tilde{g})$ be an isometry. If $\tilde{\nabla}$ is a connection on $\tilde{M}$, we define the pullback connection by

$$(g^* \tilde{\nabla})_X Y = \phi_*^{-1}(\tilde{\nabla}_{\phi_* X}(\phi_* Y)).$$

Show that $g^* \tilde{\nabla}$ is a connection on M and that if $\tilde{\nabla}$ is the Levi-Civita connection on $\tilde{M}$, then $g^* \tilde{\nabla}$ is the Levi-Civita connection on M.