Universität Tübingen, SS 2018
Topics in Riemannian Geometry
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Problem Set 2

Problem 2.1 – Submanifolds

Let (M, g) be a Riemannian manifold and let ∇ be the Levi-Civita connection on M. We suppose that $N \subset M$ is an embedded submanifold equipped with the Riemannian metric inherited from M.

a) Show that every $X \in \mathfrak{X}(N)$ has an extension $\tilde{X} \in \mathfrak{X}(U)$ where $U \subset M$ is an open set such that $N \subset U$.

b) Let $P : TM \to TN$ be given by g-orthogonal projection and define $\tilde{\nabla} : \mathfrak{X}(N) \times \mathfrak{X}(N) \to \mathfrak{X}(N)$ on N by
$$\tilde{\nabla}_X Y = P \nabla_{\tilde{X}} \tilde{Y},$$
where $\tilde{X}, \tilde{Y}$ are arbitrary extensions of X, Y. Show that $\tilde{\nabla}$ is well-defined and that it is the Levi-Civita connection on N.

Problem 2.2 – Isometries II

a) Let $\phi : (M, g) \to (\tilde{M}, \tilde{g})$ be an isometry. Show that if $\gamma : I \to M$ is a geodesic, then $\phi \circ \gamma : I \to \tilde{M}$ is also a geodesic.

b) Consider the n-sphere $S^n = \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\}$. Show that for every orthogonal $(n+1)\times(n+1)$-matrix A, the map $x \mapsto Ax$ is an isometry of S^n.

c) Consider the hyperbolic plane $H = \{x + iy \in \mathbb{C} \mid y > 0\}$ equipped with the metric $g = (dx^2 + dy^2)/y^2$. Show that map
$$z \mapsto \frac{az + b}{cz + d}$$
for $a, b, c, d \in \mathbb{R}$ is an isometry if $ad - bc = 1$.

Problem 2.3 – Laplacian

Let (M, g) be an oriented Riemannian manifold of dimension n, and let dV be the associated volume form. Recall that the Hodge star operator on k-forms, $k = 0, 1, \dots, n$ is the map $\star : \Omega^k(M) \to \Omega^{n-k}(M)$ defined by the condition that $\omega \wedge \star\eta = g(\omega, \eta)\mathrm{dV}$ for all $\omega, \eta \in \Omega^k(M)$. We now define the Laplacian Δ on functions by the formula
$$\Delta f = - \star d \star df.$$

a) Show that for $\mathbb{R}^n$ with the standard metric, we have $\Delta f = -\sum_{i=1}^n \partial_i^2 f$.

b) Show that in general, Δ is given in local coordinates by
$$-\frac{1}{\sqrt{\det g}}\, \partial_j \left(g^{ij} \sqrt{\det g}\, \partial_i f \right).$$