Problem Set 4

Problem 4.1 – Sectional curvature

Let (M, g) be a Riemannian manifold. Recall that the curvature R is defined as the mapping that maps pairs $(X, Y) \in \mathfrak{X}(M) \times \mathfrak{X}(M)$ to the map $R(X, Y) : \mathfrak{X}(M) \to \mathfrak{X}(M)$ given by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

Given two linearly independent tangent vectors $u, v \in T_pM$, we define

$$K(u, v) = \frac{g(R(u, v)v, u)}{|u|^2 |v|^2 - g(u, v)^2}.$$

Show that if $\{u, v\}$ and $\{x, y\}$ are bases of the same two-dimensional subspace of T_pM, then $K(u, v) = K(x, y)$.

Problem 4.2 – Constant sectional curvature

a) Consider $\mathbb{R}^n$ with the usual Euclidean metric $g = \sum_i (dx^i)^2$. Show that $K \equiv 0$.

b) Consider the n-sphere of radius R, $R < 0$, given by

$$S^n_R = \{x \in \mathbb{R}^{n+1} \mid \|x\| = R\},$$

equipped with the metric inherited from $\mathbb{R}^{n+1}$. Show that $K \equiv R^{-2}$.

c) Consider hyperbolic space of radius R, $R > 0$, given by

$$H^n_R = \{x \in \mathbb{R}^n \mid x^n > 0\},$$

equipped with the metric $h = R^2 (x^n)^{-2} \sum_i (dx^i)^2$. Show that $K \equiv -R^{-2}$.