Let (M, g) be a Riemannian manifold. Recall that the Ricci curvature, Ric, is the 2-tensor defined as the trace of the curvature endormorphism on its first and last indices, i.e., the components of Rc are

$$R_{ij} = R^k_{kij} = g^{km} R_{kijm}.$$

The scalar curvature, S, is the trace of the Ricci curvature, i.e.,

$$S = R^i_i = g^{ij} R_{ij}.$$

Problem 5.1 – Constant sectional curvature

Suppose that (M, g) is a Riemannian manifold of dimension n with constant curvature K. Show that

$$R(X, Y)Z = K\left(g(Y, Z)X - g(X, Z)Y\right),$$

$$R(X, Y, Z, W) = K\left(g(X, W)g(Y, Z) - g(X, Z)g(Y, W)\right),$$

$$\mathrm{Ric} = (n - 1)Kg,$$

$$S = n(n - 1)K.$$

Problem 5.2 – Second Bianchi Identity

Recall that if T is a k-tensor, then ∇T is the $(k + 1)$-tensor such that

$$\nabla T(X_1, \ldots, X_k, Y) = YT(X_1, \ldots, X_k) - T(\nabla_Y X_1, X_2, \ldots, X_k) - \cdots - T(X_1, \ldots, X_{k-1}, \nabla_Y X_k).$$

Prove that

$$\nabla R(X, Y, Z, W, T) + \nabla R(X, Y, W, T, Z) + \nabla R(X, Y, T, Z, W) = 0.$$

Problem 5.3 – Schur's Theorem

Let (M, g) be a Riemannian manifold of dimension $n \geq 3$. Suppose that M is isotropic, i.e., for each $p \in M$, the sectional curvature K_p is constant. Prove that M has constant sectional curvature, i.e., that K_p does not depend on p.