Problem 7.1 – Normal Jacobi fields

Let M be a Riemannian manifold of dimension n. Let $\gamma : I \to M$ be a geodesic and fix $a \in I$. Show that a Jacobi field J along γ is normal if and only if $J(a) \perp \dot\gamma(a)$ and $D_t J(a) \perp \dot\gamma(a)$, and conclude that the set of normal Jacobi fields along γ is a vector space of dimension $2n - 2$.

Problem 7.2 – Covariant derivative

Let $\gamma : [0, a] \to M$ be a geodesic and let $X \in \mathfrak{X}(M)$ be such that $X(\gamma(0)) = 0$. Show that

$$\nabla_{\dot\gamma} \left(R(X, \dot\gamma)\dot\gamma\right)(0) = \left(R(D_t X, \dot\gamma)\dot\gamma\right)(0)$$

. Recall that $\nabla_{\dot\gamma} R(X, Y, Z, W) = \nabla R(X, Y, Z, W, \dot\gamma)$ and use the fact that $0 = \nabla_{\dot\gamma} R(X, \dot\gamma, \dot\gamma, Z)$ for all Z and for $t = 0$.

Problem 7.3 – Conjugate points and negative curvature

Let $\gamma : [0, a] \to M$ be a geodesic. We say that $\gamma(a)$ is conjugate to $\gamma(0)$ along γ if there exists a Jacobi field J along γ, not identically zero, such that $J(0) = 0$ and $J(a) = 0$. Suppose that M has non-positive sectional curvature. Show that for every point $p \in M$, there are no points conjugate to p.