Universität Tübingen, SS 2018
Topics in Riemannian Geometry
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Problem Set 8

Problem 8.1 – Boundary value problem for Jacobi fields

Let M be a Riemannian manifold and let $\gamma : [a,b] \to M$ be a geodesic from $p = \gamma(a)$ to $q = \gamma(b)$. Show that for all $v \in T_pM$ and $w \in T_qM$ there is a unique Jacobi field such that $J(a) = v$ and $J(b) = w$ if and only if p and q are not conjugate along γ.

Problem 8.2 – Metric in normal coordinates

Let (M, g) be a Riemannian manifold and $p \in M$. Show that the second-order Taylor series of g in normal coordinates centred at p is

$$g_{ij}(x) = \delta_{ij} - \frac{1}{3} \sum_{kl} R_{iklj} x^k x^l + O(|x|^3).$$

(Hint: consider the Jacobi fields $tW^i \partial_{x^i}$ along the geodesic $t \mapsto (tV^1, \dots, tV^n)$.)

Problem 8.3 – Energy functional

Let (M, g) be a Riemannian manifold. Given a smooth curve $\gamma : [a,b] \to M$, we define the energy $E(\gamma)$ to be

$$E(\gamma) = \int_a^b g(\dot\gamma(t), \dot\gamma(t))\, dt.$$

a) Show that $\ell(\gamma)^2 \le (b-a)E(\gamma)$. When does equality hold?

b) Suppose that γ is a minimizing geodesic. Show that, for any curve $\eta : [a,b] \to M$ joining $\gamma(0)$ to $\gamma(1)$, $E(\gamma) \le E(\eta)$, and that equality holds if and only if η is a minimizing geodesic.