WPS 01: Birdtracks, symmetrizers and antisymmetrizers

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J. Alcock-Zeilinger

Exercise 1.

1. Consider the action of group S_4 on $V^{\otimes 4}$ (or, equivalently, the ordered set $\{1,2,3,4\}$). Write down the quantities $\boldsymbol{S}_{24}$ and $\boldsymbol{A}_{134}$ in birdtrack notation, using the shorthand notation (think box over certain index lines) discussed in class. Show that both these quantities are idempotent by writing them as a sum of permutations.

2. Consider the action of the group S_n on an ordered set $\{1,2,\ldots,n\}$. Let $\{a_1,a_2,\ldots a_k\}$ be a subset of $\{1,2,\ldots,n\}$. Show that the antisymmetrizer $\boldsymbol{A}_{a_1a_2\ldots a_k}$ is idempotent:

   (a) First, write $\boldsymbol{A}_{a_1a_2\ldots a_k}$ as a sum of permutations.

   (b) When forming the product $\boldsymbol{A}_{a_1a_2\ldots a_k}\cdot\boldsymbol{A}_{a_1a_2\ldots a_k}$, use the fact that the permutations in the sum of part 2a form a subgroup of S_n to argue which elements can appear in the product $\boldsymbol{A}_{a_1a_2\ldots a_k}\cdot\boldsymbol{A}_{a_1a_2\ldots a_k}$ and how often they can appear there.

   (c) Show that, for any $\rho,\sigma\in S_n$, $\mathrm{sign}(\rho)\mathrm{sign}(\sigma)=\mathrm{sign}(\rho\sigma)$.

   (d) Put everything together to obain the desired result.

Exercise 2.
Consider the action of S_3 on $V^{\otimes 3}$ (or, equivalently, the ordered set $\{1,2,3\}$). Using birdtrack notation, explicitly check that $\boldsymbol{S}_{24}=(34)\boldsymbol{S}_{23}(34)^{-1}$ by writing the quantities on either side of the equal sign as sums of permutations and comparing the outcome.

Exercise 3.
Consider the action of the group S_4 on $V^{\otimes 4}$. Knowing the definition of a trace in birdtrack notation, and using the scalar product $\langle A|B\rangle := \mathrm{tr}\left(A^\dagger B\right)$ (where $A,B\in\mathrm{Lin}\left(V^{\otimes m}\right)$), compute the following quantities:

1. $\langle(12)(34)+(243)|(134)-(24)\rangle$
2. $\langle(1234)|\boldsymbol{S}_{23}\rangle$
3. $\langle\boldsymbol{S}_{12}|\boldsymbol{A}_{23}\rangle$

Give a diagrammatic argument why the trace as defined in birdtrack notation is cyclic.

Exercise 4.
Consider the action of the group S_n on $V^{\otimes n}$, and recall the definition of a trace in birdtrack notation. Compute the following quantities:

1. $\mathrm{tr}\left(\boldsymbol{S}_{12}\right)$ and $\mathrm{tr}\left(\boldsymbol{A}_{12}\right)$
2. $\mathrm{tr}\left(\boldsymbol{S}_{123}\right)$ and $\mathrm{tr}\left(\boldsymbol{A}_{123}\right)$

Can you guess a general formula for $\mathrm{tr}\left(\boldsymbol{S}_{12\ldots n}\right)$ and $\mathrm{tr}\left(\boldsymbol{A}_{12\ldots n}\right)$?