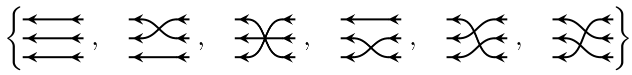
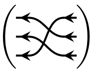
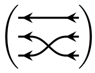
WPS 02: Group representation theory and more birdtracks

May 4$^{\text{th}}$, 2018

J. Alcock-Zeilinger

Exercise 1.
Let $\mathcal{R} : S_n \to \mathsf{GL}(\mathbb{C}, n!)$ be the left regular representation of the group S_n defined in class. For $n = 3$, consider $\widehat{S_3}$ to be the ordered set

$\widehat{S_3} :=$  . (1)

Compute the matrices $\mathcal{R}$  and $\mathcal{R}$ .

Exercise 2.
In lectures, we discussed *Maschke's Theorem*, which states that for any group G with representation $\varphi : \mathsf{G} \to \mathrm{End}(V)$, and $W \subset V$ carrying a subrepresentation of G, we can always write $V = W \oplus U$ such that U also carries a subrepresentation of G. In the proof, we defined the operator $T : V \to V$ as

$$T(v) = \frac{1}{|G|} \sum_{\mathsf{g} \in \mathsf{G}} \varphi(\mathsf{g}^{-1}) \left[\pi\left(\varphi(\mathsf{g})(v)\right)\right] , \qquad \text{for every } v \in V , \tag{2}$$

where π is a projection from V onto its subspace W, $\pi : V \to W$. Prove the following two properties of T:

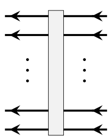
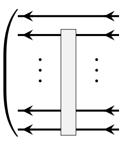
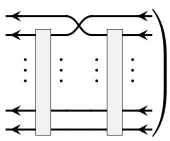
1. Show that T is idempotent on V, that is $T^2(v) = T(v)$ for every $v \in V$.
2. Show that T is invariant under φ, that is $\varphi(\mathsf{h})\left(T(v)\right) = T\left(\varphi(\mathsf{h})(v)\right)$ for every $\mathsf{h} \in \mathsf{G}$ and every $v \in V$. (*Hint:* show that the action of φ effects a reordering of the sum in T.)

Exercise 3.
In your previous problem set, you were asked to guess a formula for the trace of a symmetrizer $\boldsymbol{S}_{12\ldots k}$ acting on $V^{\otimes k}$, and you (hopefully) guessed that

$$\mathrm{tr}\left(\boldsymbol{S}_{12\ldots k}\right) = \frac{(N+k-1)!}{(N-1)!k!} , \tag{3}$$

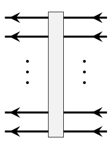
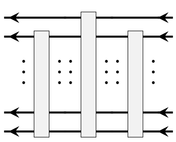
where $N = \dim(V)$. Let us prove this formula and, in the process, obtain some important intermediate results:

1. Denoting the permutation (12) by $\sigma(12)$ (in order not to cause confusion), show that $\boldsymbol{S}_{123\ldots k} = \frac{1}{k}\left(\boldsymbol{S}_{23\ldots k} + (k-1)\boldsymbol{S}_{23\ldots k}\sigma(12)\boldsymbol{S}_{23\ldots k}\right)$, that is

    $= \frac{1}{k}\Bigg($  $+ (k-1)$  $\Bigg)$, (4)

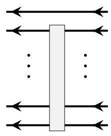
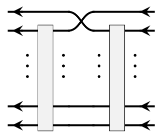
   in the following way:

   (a) Write the symmetrizer $\boldsymbol{S}_{123\ldots k}$ as a product $\boldsymbol{S}_{23\ldots k}\boldsymbol{S}_{123\ldots k}\boldsymbol{S}_{23\ldots k}$,

    $=$  (5a)

   (recall the "inclusion property" of symmetrizers from class).

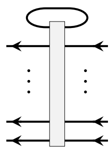
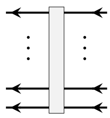
(b) Recall that the longest symmetrizer $\boldsymbol{S}_{123\ldots k}$ in eq. (5a) is a sum of permutations over the ordered set $\{1, 2, 3, \ldots, k\}$, and argue that each permutation in this sum can be absorbed in the outer two symmetrizers $\boldsymbol{S}_{23\ldots k}$ to yield either

 or  . (5b)

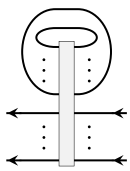
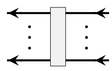
(*Hint:* It may be useful to recall that every permutation can be written as a product of transpositions.)

(c) Pay close attention to how many of these permutations will give rise to either of the factors depicted in eq. (5b) in order to obtain the desired formula (4).

2. Using formula (4), show that tracing only the first index of the symmetrizer $\boldsymbol{S}_{123\ldots k}$ yields

 $= \dfrac{N+k-1}{k}$  . (6)

3. By induction on eq. (6), show that the trace over the first p indices of $\boldsymbol{S}_{123\ldots k}$ gives

 $= \dfrac{(N+k-1)!(k-p)!}{(N+k-p-1)!k!}$  (7)

(note that $(k-p)$ index lines remained uncontracted).

4. Letting $p = k$, we obtain the desired result.

Both eq. (4) and (7) are incredibly handy in practical calculations and are therefore well worth remembering. Can you guess the corresponding formulae for the antisymmetrizer $\boldsymbol{A}_{123\ldots k}$?