WPS 03: Representations of the group algebra & idempotents

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J. Alcock-Zeilinger

Exercise 1.
Let $\mathcal{A}$ be an algebra over a field $\mathbb{F}$, let $\mathcal{M}$ be an $\mathcal{A}$-module and for each $a \in \mathcal{A}$ define a map $\phi_a : \mathcal{M} \to \mathcal{M}$ by

$$\phi_a(m) = am \qquad \text{for every } m \in \mathcal{M} \; . \tag{1}$$

1. Show that the map

$$\begin{aligned} \phi : \mathcal{A} &\to \mathrm{End}(\mathcal{M}) \\ a &\mapsto \phi_a \end{aligned} \tag{2a}$$

   is a representation of $\mathcal{A}$. [*Hint:* Show that ϕ satisfies

   (a) $\phi(a + b) = \phi(a) + \phi(b)$
   (b) $\phi(\lambda a) = \lambda\phi(a)$
   (c) $\phi(ab) = \phi(a) \circ \phi(b)$

   for every $a, b \in \mathcal{A}$ and every $\lambda \in \mathbb{F}$. Pay particular close attention to exactly *why* each step in your reasoning holds.]

2. Let $\phi : \mathcal{A} \to \mathrm{End}(V)$ be a representation of $\mathcal{A}$ on some vector space V, and define a map $\tilde{\phi} : \mathcal{A} \times V \to V$ by

$$\tilde{\phi}\big((a, v)\big) = \phi(a)(v) \tag{2b}$$

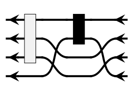
   for every $a \in \mathcal{A}$ and every $v \in V$. Show that the map $\tilde{\phi}$ endows V with an $\mathcal{A}$-module structure, allowing one to view V as a left $\mathcal{A}$-module. [*Hint:* Show that, for every $a, b \in \mathcal{A}$, every $v, w \in V$ and every $\lambda \in \mathbb{F}$ we have that

   (a) $\tilde{\phi}\big((a + b, v)\big) = \tilde{\phi}\big((a, v)\big) + \tilde{\phi}\big((b, v)\big)$
   (b) $\tilde{\phi}\big((a, v + w)\big) = \tilde{\phi}\big((a, v)\big) + \tilde{\phi}\big((a, w)\big)$
   (c) $\tilde{\phi}\big((\lambda a, v)\big) = \lambda\tilde{\phi}\big((a, v)\big)$
   (d) $\tilde{\phi}\big((ab, v)\big) = \tilde{\phi}\Big(\big(a, \tilde{\phi}\big((b, v)\big)\big)\Big)$.]

The point of this exercise is to make yourself aware in which space you are currently working, and which rules hold in this space. Therefore, firstly, understand why the properties of the maps given in the hints are indeed the properties you need to prove, and, secondly, make sure that at each step of your reasoning you fully understand why the necessary algebraic manipulations can be performed.

Exercise 2.
Using birdtrack notation, show that the operator

$$\boldsymbol{S}_{123}\boldsymbol{A}_{14} = \tag{3}$$



is quasi-idempotent: Do this by first writing the operator as a sum of permutations, and then form the product $\boldsymbol{S}_{123}\boldsymbol{A}_{14} \cdot \boldsymbol{S}_{123}\boldsymbol{A}_{14}$. Which constant α is needed to make $\alpha\boldsymbol{S}_{123}\boldsymbol{A}_{14}$ idempotent?

Exercise 3.
Let G be a group and let x be an element of G. In lectures, we defined the conjugacy class x^{G} to be

$$\mathsf{x}^{\mathsf{G}} := \left\{\mathsf{g} \in \mathsf{G} \middle| \mathsf{g} = \mathsf{h}\mathsf{x}\mathsf{h}^{-1} \text{ for some } \mathsf{h} \in \mathsf{G}\right\} \ . \tag{4}$$

Consider the symmetric group S_n. Prove that two elements $\rho, \sigma \in S_n$ are in the same conjugacy class if and only if they have the same cycle structure. To accomplish this, proceed in the following fashion:

1. First, show that for a particular k-cycle $(i_1 i_2 \ldots i_k) \in S_n$, we have that

   $$\tau\big(i_1 i_2 \ldots i_k\big)\tau^{-1} = \big(\tau(i_1)\tau(i_2)\ldots\tau(i_k)\big) \qquad \text{for every } \tau \in S_n \ . \tag{5a}$$

   Use this equation to reason that the cycle structure of a permutation $\rho \in S_n$ stays unchanged when it is conjugated by an element $\tau \in S_n$.

2. Suppose that $\rho, \sigma \in S_n$ have the same cycle structure, and show that one can always find a permutation $\tau \in S_n$ such that

   $$\rho = \tau\sigma\tau^{-1} \ . \tag{5b}$$

   (*Hint:* List ρ and σ in two lines, on above the other, taking care to always put cycles of the same length above each other. Interpret the result as a 2-line notation (*c.f.* remark below and eq. (6)) of a permutation, and ponder about which role this permutation might play in the desired outcome.)

Remark: Any permutation $\rho \in S_n$ can be written in 2-line notation as

$$\rho = \begin{pmatrix} i_1 & i_2 & i_3 & \ldots & i_n \\ \rho(i_1) & \rho(i_2) & \rho(i_3) & \ldots & \rho(i_n) \end{pmatrix} \ . \tag{6}$$