WPS 04: Everything Young: diagrams, tableaux and projection operators

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Exercise 1.
An exercise on Young diagrams an Young tableaux:

- Write down all Young *diagrams* of size 6 (i.e. consisting of six boxes).
- For each of these Young diagrams $\mathbf{Y}$ of size 6, compute the hook length $\mathscr{H}_{\mathbf{Y}}$.
- With this information, find the number of Young *tableaux* of size 6, i.e. compute $|\mathcal{Y}_6|$. [*Hint*: For a particular Young diagram $\mathbf{Y}$ consisting of n boxes, recall that the number of Young tableaux with shape $\mathbf{Y}$ is given by $\frac{n!}{\mathscr{H}_{\mathbf{Y}}}$.] By the way, you should find that $|\mathcal{Y}_6| = 76$.

Notice that, if you were only interested in the number of Young tableaux of size n, going this route via the Young diagrams and the hook lengths is not the easiest/quickest way to go, since there is not closed form exact formula for the number of Young diagrams of a certain size. Luckily however, there exists a closed form formula for the number of Young tableaux, but that is a story for another day....

Exercise 2.
Construct all Young projection operators of $\mathbb{C}[S_3]$ (acting on $V^{\otimes 3}$) in birdtrack notation:

1. Write down all Young tableaux in $\mathcal{Y}_3$ (the set of Young tableaux consisting of three boxes).
2. For each Young tableau $\Theta \in \mathcal{Y}_3$, construct the Young operator $e_\Theta := \mathbf{S}_\Theta \mathbf{A}_\Theta$.
3. To turn each Young operator e_Θ into a Young *projection* operator $Y_\Theta = \alpha_\Theta e_\Theta$, compute the required renormalization constant α_Θ (recall from lectures that α_Θ involves the hook length of Θ).

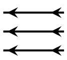
Exercise 3.
Verify the following properties of the Young projection operators of $\mathbb{C}[S_3]$ in birdtrack notation:

1. Show that, for every pair of tableaux $\Theta, \Phi \in \mathcal{Y}_3$, we have that

   $$Y_\Theta Y_\Phi = \delta_{\Theta\Phi} Y_\Theta \; ; \qquad (1)$$

   in other words, show that the Young projection operators of $\mathbb{C}[S_3]$ are idempotent and pairwise orthogonal. [*Hint:* When showing this in the birdtrack formalism, it may be useful to recall that an operator is automatically zero if two index lines of a particular antisymmetrizer are connected to the same symmetrizer.]

2. Show that the Young projection operators of $\Theta, \Phi \in \mathcal{Y}_3$ sum up to unity,

   $$\sum_{\Theta \in \mathcal{Y}_3} Y_\Theta \; = $$  . (2)

A word of caution: the pairwise orthogonality of Young projection operators breaks down for tableaux of size ≥ 4. However, the idempotency property remains for tableaux of all sizes.