WPS 05: $\mathsf{GL}(N)$, $\mathsf{SU}(N)$ and irreducible representations

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Exercise 1.
Recall the definition of the *special linear group* on a vector space V with $\dim(V) = N$, $\mathsf{GL}(N)$, to be

$$\mathsf{GL}(N) := \left\{ f \in \mathrm{End}(V) \;\middle|\; \exists f^{-1} \in \mathrm{End}(V) \ : \ ff^{-1} = \mathbb{1}_V = f^{-1}f \right\} \ , \tag{1}$$

where $\mathbb{1}_V$ is the identity map on V.

1. Show that $\mathsf{GL}(N)$ is indeed a group by checking that it contains the identity, every element has an inverse, it is closed under "multiplication" (i.e. composition of linear maps).

We briefly also talked about the *special unitary group* $\mathsf{SU}(N)$, which is defined as

$$\mathsf{SU}(N) := \left\{ U \in \mathsf{GL}(N) \;\middle|\; UU^\dagger = \mathbb{1} = U^\dagger U \text{ and } \det U = 1 \right\} \ ; \tag{2}$$

here, $U^\dagger$ denotes the Hermitian conjugate of U with respect to the canonical scalar product on V and det denotes the matrix determinant of U (recall that we may view the elements of $\mathrm{End}(V)$ as matrices). Show that $\mathsf{SU}(N)$ is also a group in the following way:

2. Consider the map $\det : \mathsf{GL}(N) \to \mathbb{C}$, where $\mathbb{C}$ is viewed as a commutative groups (with respect to multiplication) with multiplicative identity 1. Reason that this is a group homomorphism.
3. Since det is a group homomorphism, $\ker\det$ is a subgroup of $\mathsf{GL}(N)$, called the *special linear group* $\mathsf{SL}(N)$. (Note that the term *special* refers to the fact that all its elements have determinant 1).
4. Show that the subset of $\mathsf{SL}(N)$ that consists only of unitary elements is closed, thus making it a subgroup of $\mathsf{SL}(N)$. This subgroup is in fact the special unitary group $\mathsf{SU}(N)$.

Exercise 2.
Let $\mathcal{Y}_n$ be the set of all Young tableaux consisting of n boxes, and let Y_Θ be the Young projection operator corresponding to $\Theta \in \mathcal{Y}_n$. Consider the direct sum of all Young projection operators

$$\bigoplus_{\Theta\in\mathcal{Y}_n} Y_\Theta \ , \tag{3a}$$

which acts on the whole space $V^{\otimes n}$ and can therefore be visualized as a matrix of size

$$\dim(V^{\otimes n}) \times \dim(V^{\otimes n}) = N^n \times N^n \ . \tag{3b}$$

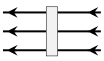
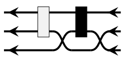
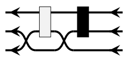
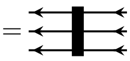
In lectures, we discussed that the Young projection operators generate the irreducible representations of $\mathsf{SU}(N)$ on $V^{\otimes n}$. That is, each Young projector Y_Θ projects onto an irreducible subspace of $V^{\otimes n}$. Thus, the matrix (3a) block-diagonalizes, and each block corresponding to a particular Y_Θ is of size $\dim(Y_\Theta) \times \dim(Y_\Theta)$. We can choose a particular basis on $V^{\otimes n}$ such that the block corresponding to Y_Θ for a particular $\Theta \in \mathcal{Y}_n$ is given by the identity matrix of size $\dim(Y_\Theta) \times \dim(Y_\Theta)$ (this is due to the fact that Y_Θ acts as the identiy on the subspace onto which it projects). Thus, the dimension of the representation corresponding to Y_Θ is merely given by $\mathrm{tr}\,(Y_\Theta)$,

$$\mathrm{tr}\,(Y_\Theta) = \mathrm{tr}\left(\underbrace{\begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}}_{\dim(Y_\Theta)\times\dim(Y_\Theta)} \right) = \sum_{i=1}^{\dim(Y_\Theta)} 1 = \dim(Y_\Theta) \ . \tag{4}$$

However, since the trace of a matrix does not depend on the choice of basis, it follows that, in general

$$\mathrm{tr}\,(Y_\Theta) = \dim(Y_\Theta)\ . \tag{5}$$

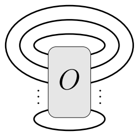
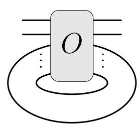
Consider the Young projection operators of $\mathsf{SU}(N)$ on $V^{\otimes 3}$ given by

$Y_{\begin{smallmatrix}1&2&3\end{smallmatrix}} =$  , $\quad Y_{\begin{smallmatrix}1&2\\3\end{smallmatrix}} = \frac{4}{3}$  , $\quad Y_{\begin{smallmatrix}1&3\\2\end{smallmatrix}} = \frac{4}{3}$  , $\quad Y_{\begin{smallmatrix}1\\2\\3\end{smallmatrix}} =$  . (6)

Calculate the dimension of the irreducible representation corresponding to each of these Young projection operators in the birdtrack formalism. [*Hint:* Recall that the trace of a birdtrack is formed by connecting the index lines on the same level, and use the formula for the partial traces of symnmetrizers and antisymmetrizers provn in WPS 02.]

Since these Young projection operators divide the space $V^{\otimes 3}$ into irreducible subspaces, the dimensions of each of the operators (i.e. the dimensions of each of the irreducible subspaces corresponding to the operators) must sum up to $\dim(V^{\otimes 3}) = N^3$. Check that this is indeed the case here.

Exercise 3.
Recall that the trace of a birdtrack operator O, $\mathrm{tr}\,(O)$, is formed by connecting all index lines on the same level. We define a partial trace $\mathrm{tr}^k(O)$ to be the trace of the *bottom k indices* of O, that is,

$\mathrm{tr}\,(O) \;=$  and $\mathrm{tr}^k(O) \;=$  . (7)

Let $\Theta \in \mathcal{Y}_n$ be a Young tableau, and let $\Theta_{(1)} \in \mathcal{Y}_{n-1}$ be the Young tableau obtained from Θ by removing the box $\boxed{n}$. If Y_Θ is the Young projection operator corresponding to Θ, it can be shown that

$$\mathrm{tr}^1(Y_\Theta) \propto Y_{\Theta_{(1)}}\ . \tag{8}$$

Explicitly verify eq. (8) for the Young projection operators Y_Θ for $\Theta \in \mathcal{Y}_4$.

Exercise 4.
Consider a 3-dimensional vector space V with basis $\{v_1, v_2, v_3\}$. Forming the tensor product space $V^{\otimes 3}$, the basis of V induces a basis on $V^{\otimes 3}$, where each basis vector of $V^{\otimes 3}$ is of the form

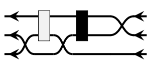
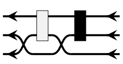
$$v_i \otimes v_j \otimes v_k \qquad \text{for } i, j, k \in \{1, 2, 3\}\ ; \tag{9a}$$

clearly, this basis has size $3^3 = 27$. (In general, if $\dim(V) = N$, the tensor product space $V^{\otimes n}$ has dimension N^n.) Introducing the shorthand notation

$$|ijk\rangle := v_i \otimes v_j \otimes v_k\ , \tag{9b}$$

the basis vectors of $V^{\otimes 3}$ are given by $|111\rangle, |112\rangle, |121\rangle, \ldots$ and so on. We will now study the irreducible representations of both S_3 and $\mathsf{SU}(3)$ on $V^{\otimes 3}$:

1. Since the irreducible representations of $\mathsf{SU}(N)$ and S_n on $V^{\otimes n}$ are generated by the Young projection operators of length n, we must again look at the Young projection operators given in eq. (6). Calculate the action of each of these Young projection operators on each of the 27 basis vectors of $V^{\otimes 3}$. [*Hint:* to make your life a lot easier, first consider various symmetries hidden within these Young projection operators; for example, you will find that

   $Y_{\begin{smallmatrix}1&3\\2\end{smallmatrix}}(12) \;=\; \frac{4}{3}$  $\;=\; -\frac{4}{3}$  $\;=\; -Y_{\begin{smallmatrix}1&3\\2\end{smallmatrix}}$ (10)

   $$\Rightarrow\quad Y_{\begin{smallmatrix}1&3\\2\end{smallmatrix}}\,|ijk\rangle \;=\; -Y_{\begin{smallmatrix}1&3\\2\end{smallmatrix}}(12)\,|ijk\rangle = -Y_{\begin{smallmatrix}1&3\\2\end{smallmatrix}}\,|jik\rangle\ , \quad \text{and hence}\quad Y_{\begin{smallmatrix}1&3\\2\end{smallmatrix}}\,|iij\rangle = 0\ .]$$

   In particular, you should find 10 nonzero, linearly independent vectors of the form $Y_{\begin{smallmatrix}1&3&2\end{smallmatrix}}\,|ijk\rangle$ ($|ijk\rangle = |ijk\rangle$ is a basis vector of $V^{\otimes 3}$), 8 nonzero, linearly independent vectors of the form $Y_{\begin{smallmatrix}1&2\\3\end{smallmatrix}}\,|ijk\rangle$, 8 nonzero, linearly independent vectors of the form $Y_{\begin{smallmatrix}1&3\\2\end{smallmatrix}}\,|ijk\rangle$, and 1 nonzero, linearly independent vector of the form $Y_{\begin{smallmatrix}1\\2\\3\end{smallmatrix}}\,|ijk\rangle$.

2. Show that, for each group element $U \in \mathsf{SU}(3)$,

$$UY_\Theta \left|ijk\right\rangle \subset Y_\Theta V^{\otimes 3} \tag{11}$$

for every basis vector $|ijk\rangle$ of $V^{\otimes 3}$. [*Hint:* use the fact that the actions of $\mathsf{SU}(3)$ and S_3 commute on $V^{\otimes 3}$.] Therefore, conclude that the Young projection operators Y_Θ give rise to $\mathsf{SU}(3)$-invariant submodules of $V^{\otimes 3}$, and therefore generate representations of $\mathsf{SU}(3)$. Recalling the number of nonzero, linearly independent vectors $Y_\Theta |ijk\rangle$ for each $\Theta \in \mathcal{Y}_3$ from part 1, argue what the dimension of each $\mathsf{SU}(3)$-invariant module $Y_\Theta V^{\otimes 3}$ (and hence each representation of $\mathsf{SU}(3)$ on $V^{\otimes 3}$) should be. Does this agree with your findings in Exercise 2 for $N = 3$?

Notice that, since the tableaux $\ytableaushort{12,3}$ and $\ytableaushort{13,2}$ have the same shape, they produce two equivalent irreducible representations of $\mathsf{SU}(N)$ — in particular, we say that the representation corresponding to either of these tableaux has multiplicity 2.

3. Consider now the action of S_3 on the vectors $Y_\Theta |ijk\rangle$ for each $\Theta \in \mathcal{Y}_3$. Convince yourself that, for a particular basis vector $\boldsymbol{v}$ of $V^{\otimes 3}$, the action of any $\rho \in S_3$ produces

$$\rho Y_\Theta \boldsymbol{v} = \sum_\Phi c_\Phi Y_\Theta \boldsymbol{v} \ , \qquad \Phi \text{ has the same shape as } \Theta,\ c_\Phi \in \mathbb{C} \ . \tag{12}$$

In other words, show that, for a fixed basis vector $\boldsymbol{v}$

$$\rho Y_{\ytableaushort{123}} \boldsymbol{v} = c_{\ytableaushort{123}} Y_{\ytableaushort{123}} \boldsymbol{v} \tag{13a}$$

$$\rho Y_{\ytableaushort{12,3}} \boldsymbol{v} = c_{\ytableaushort{12,3}} Y_{\ytableaushort{12,3}} \boldsymbol{v} + c_{\ytableaushort{13,2}} Y_{\ytableaushort{13,2}} \boldsymbol{v} \ , \qquad \rho Y_{\ytableaushort{13,2}} \boldsymbol{v} = c'_{\ytableaushort{12,3}} Y_{\ytableaushort{12,3}} \boldsymbol{v} + c'_{\ytableaushort{13,2}} Y_{\ytableaushort{13,2}} \boldsymbol{v} \tag{13b}$$

$$\rho Y_{\ytableaushort{1,2,3}} \boldsymbol{v} = c_{\ytableaushort{1,2,3}} Y_{\ytableaushort{1,2,3}} \boldsymbol{v} \ . \tag{13c}$$

Therefore, conclude that the irreducible dimension of $Y_\Theta \boldsymbol{v}$ for a fixed basis vector is given by the number of tableaux of shape Θ. What is the multiplicity of the irreducible representation $Y_\Theta \boldsymbol{v}$ for each $\Theta \in \mathcal{Y}_3$?

In this question, you should have noticed that the multiplicity and dimension of an irreducible representation generated by Y_Θ change roles when switching between representations of $\mathsf{SU}(N)$ and representations of S_n. This is in fact a general feature (not particular to hte case $n = N = 3$), but we will not prove this fact.