WPS 06: Young projectors, Hermitian Young projectors and the KS algorithm

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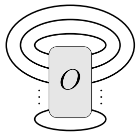
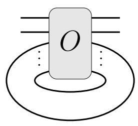
Exercise 1.
Defining the partial trace $\mathrm{tr}^k(O)$ of a birdtrack operator O to be the trace of the *bottom k indices* of O, that is,

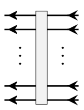
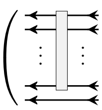
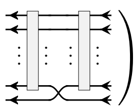
$$\mathrm{tr}\,(O) \;=$$  $\quad$ and $\quad$ $$\mathrm{tr}^k(O) \;=$$  , $\qquad (1)$

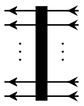
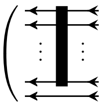
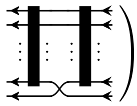
you explicitly showed in the previous problem set that $\mathrm{tr}^1(Y_\Theta) \propto Y_{\Theta_{(1)}}$ for all $\Theta \in \mathcal{Y}_3$. In this problem set, you will prove the general recursion relation: Let $\Theta \in \mathcal{Y}_n$ be a Young tableau, and suppose that the bottom symmetrizer in $\mathbf{S}_\Theta$ has length s, and the bottom antisymmetrizer in $\mathbf{A}_\Theta$ has length p (in other words, the last row of Θ has length s and the last column of Θ has length p). Then, you will show that

$$\mathrm{tr}^1(Y_\Theta) = (N+s-p)\frac{\mathscr{H}_\Theta}{\mathscr{H}_{\Theta_{(1)}}} Y_{\Theta_{(1)}} \ , \tag{2}$$

where $\mathscr{H}_\Phi$ is the hook lenght of Φ and $\Theta_{(1)}$ is the parent tableau of Θ.

1. Using the formulae

$$\boldsymbol{S}_{123\ldots s} \;=$$  $$= \frac{1}{s}\Bigg($$  $$+\ (s-1)$$  $$\Bigg) \qquad \text{(3a)}$$

$$\boldsymbol{A}_{123\ldots p} \;=$$  $$= \frac{1}{p}\Bigg($$  $$-\ (p-1)$$  $$\Bigg) \qquad \text{(3b)}$$

   decompose the bottommost symmetrizer and antisymmetrizer in the operator Y_Θ; this should yield 4 terms.

2. Thereafter, take the trace over the bottom index of each operator in the sum. Doing so, three of the four terms obtained in the previous step become proportional to $Y_{\Theta_{(1)}}$. The remaining term contains on the bottom two symmetrizers of length $s-1$ and two antisymmetrizers of length $p-1$ — call this term B_Θ.

3. Notice that B_Θ connects a symmetrizer of length $s-1$ (call this symmetrizer $\boldsymbol{S}'$) and an antisymmetrizer of length $p-1$ (call this antisymmetrizer $\boldsymbol{A}'$) via the trace over the bottom index. Use this fact to argue that B_Θ must vanish identically. [*Hint:* Notice that $\boldsymbol{S}'$ must connect so some antisymmetrizer in $\mathbf{A}_\Theta$, say to $\boldsymbol{A}^a$. This antisymmetrizer must have a common leg with some other symmetrizer $\boldsymbol{S}^b \in \mathbf{S}_\Theta$ that also connects to $\boldsymbol{A}'$. Argue that $\boldsymbol{S}'$ and $\boldsymbol{A}'$ being connected by the trace manifests in $\boldsymbol{A}^a$ and $\boldsymbol{S}^b$ receiving a *second* connecting leg, therefore yielding the operator B_Θ to vanish.]

4. You have just showed that $\mathrm{tr}^1(Y_\Theta) \propto Y_{\Theta_{(1)}}$. Provided you have not made any mistakes in part 2, it should only take minor algebraic manipulations to show that the correct proportionality constant is indeed what is given in eq. (2) [*Hint:* You should not forget that $Y_\Theta := \alpha_\Theta e_\Theta$, where e_Θ is merely a product of symmetrizers and antisymmetrizers, and α_Θ is the normalization constant containing products of lengths of the columns and rows of Θ and the hook length $\mathscr{H}_\Theta$.]

Exercise 2.

The KS algorithm discussed in class asserts that a Hermitian version of Young projection operator corresponding to a tableau $\Theta \in \mathcal{Y}_n$, P_Θ, can be recursively constructed as

$$P_\Theta := P_{\Theta_{(1)}} Y_\Theta P_{\Theta_{(1)}} \ , \tag{4}$$

where $P_{\Theta_{(1)}}$ is the Hermitian Young projection operator corresponding to hte parent tableau $\Theta_{(1)}$ canonically embedded into $\mathrm{Lin}(V^{\otimes n})$, and the stopping criterion of this recursion is given by $P_\Phi = Y_\Phi$ for $\Phi \in \mathcal{Y}_2$.

1. Using the definition of the KS projectors (4), show that P_Θ satisfies the same recursion relation regarding the partial trace (2) as their Young counterpart, that is

$$\mathrm{tr}^1(P_\Theta) = (N + s - p)\frac{\mathscr{H}_\Theta}{\mathscr{H}_{\Theta_{(1)}}} P_{\Theta_{(1)}} \ . \tag{5}$$

   [*Hint:* This is easiest shown in the birdtrack formalism.]

2. Having shown that P_Θ and Y_Θ both satisfy eq. (5), use induction to prove that

$$\mathrm{tr}\,(P_\Theta) = \mathrm{tr}\,(Y_\Theta) \tag{6}$$

   for every $\Theta \in \mathcal{Y}_n$.

Exercise 3.

Use the KS algorithm (eq. (4)) to construct all Hermitian Young projection operators corresponding to the Young tableaux in $\mathcal{Y}_4$. Use the idempotency and inclusion properties of symmetrizers and antisymmetrizers to simplify the result as much as possible.