WPS 07: KS projectors & manipulating birdtrack operators

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Exercise 1.
Consider the Young tableau

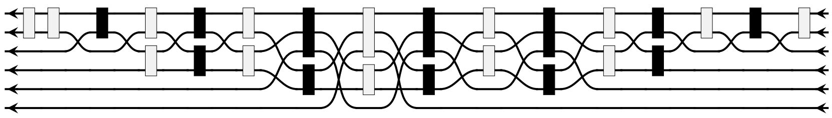
$$\Theta = \begin{array}{|c|c|c|}\hline 1 & 2 & 6 \\ \hline 3 & 4 \\ \cline{1-2} 5 \\ \cline{1-1}\end{array} \; . \tag{1}$$

Construct the KS projection operator P_Θ using the formula

$$P_\Theta = P_{\Theta_{(1)}} Y_\Theta P_{\Theta_{(1)}} \tag{2}$$

in the birdtrack formalism, and use first principles (factoring out (anti-)symmetrizers and using the idempotency of Young projectors) to simplify this operator to

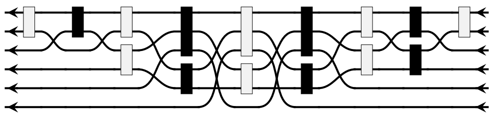
$$P_\Theta \;=\; \beta_\Theta \cdot Y_{\Theta_{(4)}} Y_{\Theta_{(3)}} Y_{\Theta_{(2)}} Y_{\Theta_{(1)}} Y_\Theta Y_{\Theta_{(1)}} Y_{\Theta_{(2)}} Y_{\Theta_{(3)}} Y_{\Theta_{(4)}}$$

$$= \beta_\Theta \cdot$$  , (3a)

where

$$\beta_\Theta := \underbrace{\alpha^2_{\Theta_{(4)}}}_{=1} \alpha^2_{\Theta_{(3)}} \alpha^2_{\Theta_{(2)}} \alpha^2_{\Theta_{(1)}} \alpha_\Theta = \frac{16384}{405} \; . \tag{3b}$$

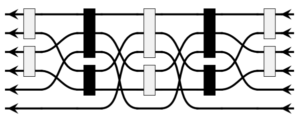
Using the simplification rules discussed in class again, show that this operator simplifies to

$$P_\Theta \;=\; \beta'_\Theta \cdot$$  (4a)

with

$$\beta'_\Theta := \alpha_{\Theta_{(3)}} \alpha_{\Theta_{(2)}} \alpha_{\Theta_{(1)}} \alpha_\Theta = \frac{2048}{45} \; . \tag{4b}$$

Just for your information: the operator (4) can bbe simplified even further to

$$P_\Theta \;=\; \frac{32}{5} \cdot$$  , (5)

which you will learn about in class shortly.

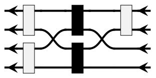
Exercise 2.
Construct the KS projection operators corresponding to the tableaux in $\mathcal{Y}_\Theta$ using the compact construction algorithm ($P_\Theta = Y_{\Theta_{(n-2)}} \cdots Y_\Theta \cdots Y_{\Theta_{(n-2)}}$).

Note that the outermost Young projector of each of these KS operators P_Θ is the one corresponding to the parent tableau $\Theta_{(1)}$. Use this fact to explicitly verify that

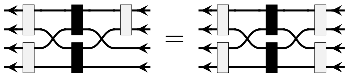
$$P_{\begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array}} + P_{\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}} = P_{\begin{array}{|c|c|}\hline 1&2\\\hline\end{array}} \qquad \text{and} \qquad P_{\begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}} + P_{\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}} = P_{\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}} \; . \tag{6}$$

You should try to accomplish this using mostly properties of symmetrizers and antisymmetrizer — you should only have to resolve at most one (anti-)symmetrizer of length two into primitive invariants for each equation.

Exercise 3.
Consider the operator

$O :=$  ; (7)

you will show that

$O :=$  (8)

in the following way:

1. First, factor a transposition out of each symmetrizer on the left.
2. Swap the top and bottom antisymmetrizer in the resulting expression, and carefully check what happens to the index lines when you do that. [*Hint:* this should "move" the transpositions to the right hand side of the antisymmetrizers.]
3. Adding up the different expressions you obtained for O should give you the desired result.

In class, we will learn the general criteria an operator needs to satisfy to be able to perform similar steps as demonstrated above. Think about what these criteria could be.