WPS 08: Simplification rules for birdtracks & amputated tableaux

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In this problem set, you should at no point have to resort to writing (anti-) symmetrizers as sums of permutations, but only use the cancellation and propagation rules discussed in class.

Exercise 1.
In WPS 06 you constructed all Hermitian Young projection operators of SU(N) on $V^{\otimes 4}$ using the KS algorithm. Take these projection operators and use the cancellation and propagation rules discussed in class to make them as short as possible and manifestly Hermitian.

Exercise 2.
In class, we discussed the concept of tableau permutations. As a reminder, for two Young tableaux $\Theta, \Phi \in \mathcal{Y}_n$ of the same shape, the tableau permutation $\rho_{\Theta\Phi}$ is defined to be the unique permutation in S_n that yields the tableau Θ when acted upon Φ,

$$\rho_{\Theta\Phi}(\Phi) = \Theta \tag{1}$$

(we understand the action of a permutation of a tableau as rearranging the numbers within the tableau as prescribed by the permutation).

1. Find the tableau permutations between all pairs of tableaux with the same shape in $\mathcal{Y}_4$.

2. For any pair of tableaux $(\Theta, \Phi) \in \mathcal{Y}_4 \times \mathcal{Y}_4$, construct the product

$$T_{\Theta\Phi} := P_\Theta \rho_{\Theta\Phi} P_\Phi \ , \tag{2}$$

where $P_{\Theta,\Phi}$ are the Hermitian Young projection operators constructed in Exercise 1, and simplify this product as much as possible using the simplification rules discussed in class. (Note that you should find 14 such transition operators.) The operator $T_{\Theta\Phi}$ is called a *transition operator* between P_Θ and P_Φ.

3. Using only the cancellation and propagation rules discussed in class, verify that the transition operators constructed in eq. (2) satisfy the following equations:

$$P_\Theta T_{\Theta\Phi} = T_{\Theta\Phi} = T_{\Theta\Phi} P_\Phi \tag{3a}$$
$$T_{\Theta\Phi} T_{\Phi\Theta} = P_\Theta \tag{3b}$$
$$T_{\Theta\Phi} = T^\dagger_{\Phi\Theta} \ . \tag{3c}$$

Exercise 3.
Under a semi-standard irregular tableau we understand an arrangement of boxes (not necessarily left-aligned or top-aligned), in which each box contains a unique integer as its entry, for example,

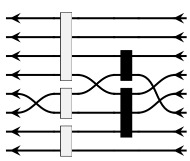
1	10	3		
		6	5	4
			7	
		9	2	8

. (4)

(Notice that every Young tableau is a semistandard irregular tableau.) For each semistandard irregular tableau $\tilde{\Theta}$, one may construct the set of symmetrizers $\mathbf{S}_{\tilde{\Theta}}$ and antisymmetrizers $\mathbf{A}_{\tilde{\Theta}}$ by symmetrizing over the rows and antisymmetrizing over the columns of the tableau, respectively, for example,

$\tilde{\Theta} =$

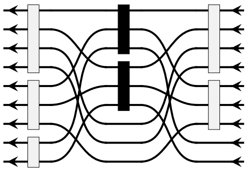
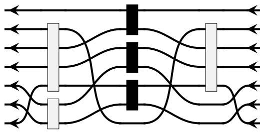
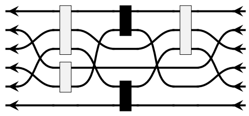
1	2	3	4	
		6	5	
			7	8

$\longrightarrow \quad \mathbf{S}_{\tilde{\Theta}} \mathbf{A}_{\tilde{\Theta}} =$  . (5)

Verify that the following birdtrack operators each are of the form

$$\mathbf{S}_{\tilde{\Theta}}\mathbf{A}_{\tilde{\Theta}}\mathbf{S}_{\tilde{\Theta}\setminus\mathcal{R}}\ , \tag{6}$$

where $\tilde{\Theta}$ is a semi-standard irregular tableau and $\mathcal{R}$ is a row in the tableau, and construct $\tilde{\Theta}$ for each of the following operators:

 ,  and  . (7)

Check whether the column amputated tableaux $\tilde{\not\Theta}_c[\mathcal{R}]$ are rectangular in each case. (You may have noticed that $\tilde{\Theta}$ is not necessarily unique, but whether the amputated tableau is rectangular or not is nonetheless a well-posed problem.)

It can be shown that, if this is the case the, the symmetrizer $\boldsymbol{S}_{\mathcal{R}}$ may be propagated to the right hand side of the operator, making it manifestly Hermitian (not necessarily idempotent though!). In other words, the propagation rules discussed in class also apply to semi-standard irregular tableaux.