WPS 09: MOLD algorithm & standard Young projectors

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J. Alcock-Zeilinger

Exercise 1.

According to the MOLD algorithm discussed in class, the Hermitian Young projection operator P_Θ corresponding to a Young tableau $\Theta \in \mathcal{Y}_n$ is given by

$$P_\Theta \propto \mathbf{I}_{\Theta_{(\mathcal{M}(\Theta))}} \; \mathbf{B}_{\Theta_{(\mathcal{M}(\Theta)-1)}} \; \ldots \begin{Bmatrix} \mathbf{B}_{\Theta_{(1)}} & \mathbf{I}_\Theta & \mathbf{B}_\Theta & \mathbf{I}_\Theta & \mathbf{B}_{\Theta_{(1)}} \\ \mathbf{I}_{\Theta_{(1)}} & \mathbf{B}_\Theta & \mathbf{I}_\Theta & \mathbf{B}_\Theta & \mathbf{I}_{\Theta_{(1)}} \end{Bmatrix} \ldots \mathbf{B}_{\Theta_{(\mathcal{M}(\Theta)-1)}} \; \mathbf{I}_{\Theta_{(\mathcal{M}(\Theta))}} \;, \tag{1}$$

where the top row applies if the MOLD $\mathcal{M}(\Theta)$ is even and the bottom row applies if $\mathcal{M}(\Theta)$ is odd, and

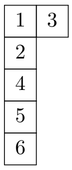
$$\Theta_{(\mathcal{M}(\Theta))} \text{ is row-ordered} \quad \Rightarrow \quad \mathbf{I} = \mathbf{S} \quad \text{and} \quad \mathbf{B} = \mathbf{A} \tag{2a}$$

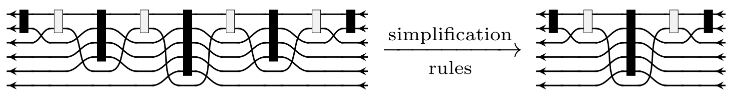
$$\Theta_{(\mathcal{M}(\Theta))} \text{ is column-ordered} \quad \Rightarrow \quad \mathbf{I} = \mathbf{A} \quad \text{and} \quad \mathbf{B} = \mathbf{S} \;. \tag{2b}$$

Using eq. (1), construct the MOLD projection operators corresponding to the Young tableaux in $\mathcal{Y}_5$ (there should be 26 such operators). For the purpose of this exercise, ignore the correct normalization constant and focus on the birdtrack part only.

Exercise 2.

Note that even though the MOLD algorithm yields much compacter birdtrack expressions for the Hermitian Young projection operators than the KS algorithm, there are still some instances where the MOLD projector can be simplified quite a lot using the propagation and cancellation rules discussed in class. An example of this would be

$\Theta :=$ 

$\Rightarrow \quad P_\Theta \propto$ 

. (3)

Can you think of some criteria that the tableau Θ needs to fulfill for the MOLD algorithm to yield the most compact form of P_Θ?

Exercise 3.

In class, we proved that the Hermitian Young projection operators (both the KS operators and the MOLD operators) satisfy the nestedness property

$$P_{\Theta_{(m)}} P_\Theta = P_\Theta = P_\Theta P_{\Theta_{(m)}} \;, \tag{4}$$

where $\Theta_{(m)}$ denotes the ancestor tableau of Θ m generations back. You will now prove that eq. (4) does not hold for the standard Young projection operators Y_Θ:

1. Show that $Y_\Theta Y_{\Theta_{(m)}} = Y_\Theta$ does not hold in general by finding a counter example. Similarlyl, foind an example for which $Y_{\Theta_{(m)}} Y_\Theta = Y_\Theta$ does not hold. [*Hint:* Since the reason why the nestedness property (4) does not hold for standard Young projectors is, essentially, their lack of Hermiticity, you should try to find an example in which both Y_Θ and $Y_{\Theta_{(m)}}$ are not Hermitian.]

2. Even though you have just shown that $Y_\Theta Y_{\Theta_{(m)}} \neq Y_\Theta$ and/or $Y_{\Theta_{(m)}} Y_\Theta \neq Y_\Theta$ for a general Young projection operator Y_Θ, one might still hope that Y_Θ at least commutes with its ancestor,

   $$\left[Y_\Theta, Y_{\Theta_{(m)}}\right] \overset{?}{=} 0 \;. \tag{5}$$

   However, also eq. (5) breaks down for a general Young projection operator:

   (a) Assume that (5) holds for a general Young projection operator Y_Θ while

   $$Y_\Theta Y_{\Theta_{(m)}} \neq Y_\Theta \qquad \text{and/or} \qquad Y_{\Theta_{(m)}} Y_\Theta \neq Y_\Theta \;. \tag{6}$$

   (b) Reformulate eq. (5) to $Y_\Theta Y_{\Theta_{(m)}} = Y_{\Theta_{(m)}} Y_\Theta$ and multiply this equation on the right (or left) with Y_Θ.

   (c) Use the cancellation rules discuss in class to arrive at a contradiction.