WPS 10: Transition operators & the algebra of invariants

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Exercise 1.
Construct all unitary transition operators $T_{\Theta\Phi}$ between equivalent MOLD projection operators P_Θ, P_Φ of $\mathsf{SU}(N)$ on $V^{\otimes 4}$ using the graphical cutting-and-gluing procedure discussed in class. Find the appropriate normalization constant by requiring

$$T_{\Theta\Phi}T_{\Phi\Theta} \stackrel{!}{=} P_\Theta \tag{1}$$

to hold for all transition operators.

Do these operators agree with your results from WPS 08 Exercise 2?

Exercise 2.
In class, we proved that the set Ω_n containing all Hermitian Young projection operators and all unitary transition operators of $\mathsf{SU}(N)$ on $V^{\otimes n}$ forms a basis for the algebra of invariants $\mathbb{C}[S_n]$. Verify this explicitly for $n = 3$ by writing each permutation in S_3 as a linear combination of the MOLD projection and transition operators on $V^{\otimes 3}$.

Exercise 3.
Let G be a finite group, and let $\varphi : \mathsf{G} \to \mathrm{End}(V)$ be a representation of G. We say that φ is a *unitary* representation if there exists a scalar product $\sigma\langle\cdot|\cdot\rangle : V \times V \to \mathbb{C}$ such that

$$\sigma\langle\varphi(\mathsf{g})\boldsymbol{v}_1|\varphi(\mathsf{g})\boldsymbol{v}_2\rangle = \sigma\langle\boldsymbol{v}_1|\boldsymbol{v}_2\rangle \tag{2}$$

for all $\mathsf{g} \in \mathsf{G}$ and for all $\boldsymbol{v}_1, \boldsymbol{v}_2 \in V$.

Consider the inner product $\sigma\langle\cdot|\cdot\rangle$ defined by

$$\sigma\langle\boldsymbol{v}_1|\boldsymbol{v}_1\rangle := \sum_{\mathsf{g}\in\mathsf{G}} \langle\varphi(\mathsf{g})\boldsymbol{v}_1|\varphi(\mathsf{g})\boldsymbol{v}_1\rangle \ , \tag{3a}$$

where $\langle\cdot|\cdot\rangle$ is *any* scalar product on V. Clearly, the inner product $\sigma\langle\cdot|\cdot\rangle$ defined in (3a) satisfies eq. (2). Therefore, we have seen that, for a finite group G, one can always find a scalar product with respect to which the representation $\varphi : \mathsf{G} \to \mathrm{End}(V)$ is unitary.

The analogous example holds for compact lie groups (such as, for example, $\mathsf{SU}(N)$), and the appropriate scalar product is given by

$$\sigma\langle\boldsymbol{v}_1|\boldsymbol{v}_1\rangle := \int_{\mathsf{G}} \langle\varphi(\mathsf{g})\boldsymbol{v}_1|\varphi(\mathsf{g})\boldsymbol{v}_1\rangle \,\mathrm{d}\mathsf{g} \ , \tag{3b}$$

where $\int_{\mathsf{G}}\mathrm{d}\mathsf{g}$ is the *Haar-integral* and $\mathrm{d}\mathsf{g}$ is called the *Haar measure* (we will not show that the inner product (3b) satisfies eq. (2) as this would require a closer study of the Haar measure, see, for example, *Groups and Symmetries — From Finite Groups to Lie Groups* by Y. Kosmann-Schwarzbach). However, we will use the result that every representation of a compact Lie group is equivalent to a unitary one (via the scalar product (3b)), and we may therefore consider every representation to be unitary.

1. Consider the group $\mathsf{SU}(N)$. Let φ be a unitary representation of $\mathsf{SU}(N)$ on $V^{\otimes n}$. Show that this implies that

   $$\varphi^\dagger(U) = \varphi(U^\dagger) \tag{4}$$

   for every $U \in \mathsf{SU}(N)$.

2. Let $\varphi_1 : \mathsf{SU}(N) \to \mathrm{End}(V_1)$ and $\varphi_2 : \mathsf{SU}(N) \to \mathrm{End}(V_2)$, $V_1, V_2 \subset V^{\otimes n}$, be two equivalent unitary representations of $\mathsf{SU}(N)$ on $V^{\otimes n}$. Then, by Schur's Lemma, there exists an $\mathsf{SU}(N)$-isomorphism $I_{12} : V_2 \to V_1$ (intertwining operator) between these two representations. Show that this $\mathsf{SU}(N)$-isomorphism must satisfy $I_{12}^\dagger = I_{21}$ (where $I_{21} := I_{12}^{-1}$). Therefore, infer that the transition operator $T_{12} := P_1 I_{12} P_2$ (where $P_{1,2}$ are the Hermitian projection operators corresponding to the representations (φ_1, V_1) and (φ_2, V_2), respectively) must also satisfy $T_{12}^\dagger = T_{21}$.