WPS 11: Various results regarding SU(N) representations

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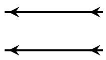
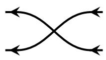
Exercise 1.
Let M be an $n \times n$ matrix, and let M^t denote its transpose. The matrix element in the i^{th} row and j^{th} column will be denoted by ${M^i}_j$ (the upper index denotes the row, the lower index denotes the column). By explicitly writing the matrix M and transposing it, show that the components of M^t satisfy

$$ {(M^t)^i}_j = {M_j}^i \; . \tag{1}$$

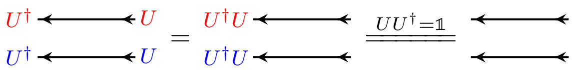
Exercise 2.
Recall that we defined a linear invariant ρ of SU(N) on a tensor product space W to be a linear map that satisfies

$$ U^\dagger \rho U = \rho \; , \tag{2}$$

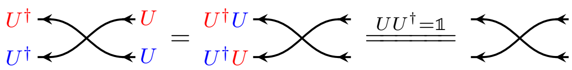
where the "multiplication" denotes the composition of linear maps on W. As you know, the primitive invariants of SU(N) on the space $V^{\otimes 2} = V \otimes V$ are given by the permutations in S_2,

$$\mathrm{id}_2 = {\delta^{j_1}}_{i_1} {\delta^{j_2}}_{i_2} =$$  and $$(12) = {\delta^{j_2}}_{i_1} {\delta^{j_1}}_{i_2} =$$  . (3)

1. Explicitly verify that these are indeed invariants of SU(N) on $V \otimes V$. [*Hint:* You need to show that each of the products of Kronecker deltas in eq. (3) satisfies $\gamma(U)^\dagger \otimes \gamma(U)^\dagger \rho \gamma(U) \otimes \gamma(U) = \rho$, where γ is the fundamental representation of SU(N) on V.]
2. Carefully go through each step in your calculation of part 1 to show that this calculation, in the birdtrack formalism, amounts to merely moving the U's along the index lines in the direction of the arrows,



(4a)

and 

. (4b)

Exercise 3.
In class, we only discussed the concept of transition operators between Hermitian Young projection operators. The reason for this is that things go wrong with "transition operators" between standard Young projection operators — this is what we are going to explore now:

Consider the — for lack of a better word — transition operator $\mathcal{T}_{\Theta\Phi}$ between two Young projection operators defined by

$$ \mathcal{T}_{\Theta\Phi} := \tau \cdot Y_\Theta \rho_{\Theta\Phi} Y_\Phi \; , \tag{5}$$

where τ is a real, non-zero constant. Show that these operators satisfy the following properties:

$$ Y_\Theta \mathcal{T}_{\Theta\Phi} = \mathcal{T}_{\Theta\Phi} = \mathcal{T}_{\Theta\Phi} Y_\Phi \tag{6a}$$
$$ \mathcal{T}_{\Theta\Phi}\mathcal{T}_{\Phi\Theta} = Y_\Theta \tag{6b}$$
$$ \mathcal{T}_{\Phi\Theta}\mathcal{T}_{\Theta\Phi} = Y_\Phi \; . \tag{6c}$$

For the unitary transition operators $T_{\Theta\Phi}$ between Hermitian Young projection operators, eq. (6b) is completely equivalent to (6c) since $T_{\Theta\Phi} = T^\dagger_{\Phi\Theta}$. Verify that

$$ \mathcal{T}_{\Theta\Phi} \stackrel{?}{=} \mathcal{T}^\dagger_{\Phi\Theta} \tag{7}$$

does not hold by using the fact that the standard Young projection operators are not Hermitian in general. Explicitly give an example where eq. (7) breaks down. [*Hint:* You should have to look no further than to the transition operators between Young projectors on $V^{\otimes 3}$.]

The fact that eq. (7) is false for all operators $\mathcal{T}_{\Theta\Phi}$ shows that $\mathcal{T}_{\Theta\Phi}$ does not constitute a SU(N)-isomorphism, since all SU(N) representations can be considered to be unitary.