NO-HIDDEN-VARIABLES THEOREMS

SCHOOL ON PARADOXES IN QUANTUM PHYSICS

JOHN BELL INSTITUTE FOR THE FOUNDATIONS OF PHYSICS

Hvar, Croatia

5 SEPTEMBER 2019

Jean BRICMONT

Institut de Recherche en Mathématique et Physique

UCLouvain

BELGIUM

A FREQUENT OBJECTION TO DE BROGLIE-BOHM's THEORY: WHAT IS SO SPECIAL ABOUT POSITIONS?

WHY NOT INTRODUCE "HIDDEN VARIABLES" FOR **ALL** OBSERVABLES?

TWO ANSWERS

1. All measurements are ultimately measurements of positions, as we saw: spin, momentum, but also energy, angular momentum etc.

   THINK of classical mechanics: everything (momentum, energy, angular momentum) is a function of the trajectories.

   So, if we recover the statistical predictions of quantum mechanics for the positions measurements (and we do thanks to quantum equilibrium), we recover the statistical predictions of quantum mechanics for **all** observables.

   No need for other "hidden variables".

2. A deeper reason: introducing "hidden variables" for **all** observables is **IMPOSSIBLE** $\rightarrow$ the goal of this talk.

In showing that, we will disprove once and for all the naive statistical interpretation of quantum mechanics as well as clarify the so-called "quantum logic".

Message of this talk in a nutshell:

"MEASUREMENTS" DON'T MEASURE AND "OBSERVATIONS" DON'T OBSERVE.

LET US GO BACK TO STANDARD QUANTUM MECHANICS.

THE GENERAL QUANTUM ALGORITHM:

$\Psi =$ STATE: VECTOR IN A HILBERT SPACE, e.g. $L^2(\mathbb{R}^N)$.

$$\Psi_0 \longrightarrow \Psi_t = U(t)\Psi_0$$

$U(t)$ UNITARY OPERATOR

$=$ SOLUTION OF SCHRÖDINGER'S EQUATION

A "OBSERVABLE" = SELF-ADJOINT OPERATOR ACTING ON THAT HILBERT SPACE.

IF A HAS A BASIS OF EIGENVECTORS:

$A\Psi_i = \lambda_i \Psi_i$

$\Psi = \sum_i c_i \Psi_i \qquad \sum_i |c_i|^2 = 1$

PROBA (Result $= \lambda_i$ when measure A, if state $= \Psi) = |c_i|^2$

AFTER THAT, THE STATE JUMPS OR IS REDUCED OR COLLAPSES TO Ψ_i.

POSSIBLE WORRY: Two mutually incompatible rules of evolution for Ψ; one about what happens "outside of measurements" and one about what happens "during measurements".

ANSWER FROM THE "NO WORRY" PHYSICISTS:

Measurements reveal pre-existing properties of the system and the quantum state gives the statistical distribution of those properties.

The collapse rule then simply comes from the fact that one changes one's probabilities after an observation (since we just learned something about the system).

It is just like coin flipping: if we don't look on which face the coin fell, we assign a probability ($\frac{1}{2}$) to each face.

Once we look on which face the coin fell, the probabilities jump to zero and one.

Nothing mysterious here.

Isn't it just like that in quantum mechanics?

POSSIBLE WORRY: Non commuting operators, interference effects.

Not the real reason!

Let us formalize this idea; for every individual system with a given quantum state Ψ, there would exist a

MAP $\quad v : \mathcal{A} \rightarrow \mathbb{R}$

so that $\forall A \in \mathcal{A}$, $v(A) =$ pre-existing, but unknown value of the "observable" A for an individual system.

Of course, we must have

$v(A) \in$ {eigenvalues of A}.

The quantum state would then give a probability distribution over those maps, following the quantum algorithm:

If the state is Ψ, and the operator A has a basis of eigenvectors,

$$A\Psi_i = \lambda_i \Psi_i$$

we write:

$$\Psi = \sum_i c_i \Psi_i \qquad \sum_i |c_i|^2 = 1$$

$$\text{PROBA } (v(A) = \lambda_i) = |c_i|^2.$$

Now, this value pre-exists to measurements and has nothing to do with them. Proper measurements will of course reveal that value but they do not create it.

This “value map” is sometimes called “non contextual” because it yields the same value for all proper measurements of A.

In particular, that value does not depend on the particular arrangement of the apparatus that is supposed to “measure observable A”, nor on whether one measures simultaneously with A an operator B that commutes with A.

NO HIDDEN VARIABLES THEOREMS: PROBLEM FOR THE STATISTICAL INTERPRETATION OR THE IMPLICIT VIEW.

(Gleason, Kochen-Specker, Bell...)

THEOREM 1:

$\nexists$ MAP $\quad v : \mathcal{A} \to \mathbb{R}$

$\mathcal{A}$= set of matrices or of operators (in a space of dimension at least equal to three), for example spin matrices and their products,

such that $\quad \forall\, A, B \in \mathcal{A}$

1) $v(A) \in$ {eigenvalues of A}

2) If $[A, B] = AB - BA = 0$, then

$$v(A + B) = v(A) + v(B)$$

THEOREM 2 (simplified version due to D. Mermin):

$\nexists$ MAP $\quad v : \mathcal{A} \rightarrow \mathbb{R}$

$\mathcal{A}$= set of matrices or of operators (in a space of dimension at least equal to four), for example spin matrices and their products,

such that $\quad \forall\ A, B \in \mathcal{A}$

1) $v(A) \in \ \{\text{eigenvalues of } A\}$

2) If $[A, B] = AB - BA = 0$, then

$$v(AB) = v(A)v(B)$$

NO USE OF THE QUANTUM FORMALISM EXCEPT FOR RULES 1 AND 2

1) $v(A) \in \{\text{eigenvalues of } A\}$

2) If $[A, B] = AB - BA = 0$, then

$$v(A + B) = v(A) + v(B)$$

or $$v(AB) = v(A)v(B),$$

THAT ARE SATISFIED BY THE QUANTUM PREDICTIONS.

THE QUANTUM STATE CANNOT GIVE A PROBABILITY DISTRIBUTION ON THINGS (THE MAPS v) THAT DO NOT EXIST!

CONCLUSION :

THE STATISTICAL VIEW (WHICH IS THE MOST NATURAL ONE) IS UNTENABLE.

THE STATISTICAL VIEW IS SOMETIMES CALLED “NAIVE REALISM WITH RESPECT TO OPERATORS” (NRAO)

MEASUREMENTS DO NOT SIMPLY “MEASURE”. THEY IN SOME SENSE ACT ON THE SYSTEM.

BUT HOW? IN ORDINARY QUANTUM MECHANICS, THEY ARE A DEUS EX MACHINA.

ONLY A MORE DETAILED THEORY CAN EXPLAIN HOW THEY ACT.

PROOF OF THEOREM 1 (outline)

To prove the theorem, we do not need to assume that the map is defined on all operators in $\mathcal{A}$.

The Kochen and Specker proof uses the squares of spin matrices, S_x, S_y, S_z, for spin associated to any three dimensional set of orthogonal vectors x, y, z in $\mathbb{R}^3$. They have the following properties:

1. The eigenvalues of S_x^2, S_y^2 and S_z^2 are 0 and 1.

2. $[S_x^2, S_y^2] = [S_y^2, S_z^2] = [S_z^2, S_x^2] = 0$.

3. $S_x^2 + S_y^2 + S_z^2 = 2$.

1. The eigenvalues of S_x^2, S_y^2 and S_z^2 are 0 and 1.

2. $[S_x^2, S_y^2] = [S_y^2, S_z^2] = [S_z^2, S_x^2] = 0$.

3. $S_x^2 + S_y^2 + S_z^2 = 2$.

From that and the assumptions:

1. $v(A) \in \{\text{eigenvalues of } A\}$.

2. If $[A, B] = AB - BA = 0$, then

$$v(A + B) = v(A) + v(B),$$

it follows that the triple $(v(S_x^2), v(S_y^2), v(S_z^2))$ must be either $(1, 1, 0)$ or $(1, 0, 1)$ or $(0, 1, 1)$.

So, the triple $(v(S_x^2), v(S_y^2), v(S_z^2))$ must be either $(1,1,0)$ or $(1,0,1)$ or $(0,1,1)$.

But that must hold *for every* set of three dimensional orthogonal vectors x, y, z in $\mathbb{R}^3$. Kochen and Specker were able to exhibit a finite number of such sets so that the above assumption on the values taken by $(v(S_x^2), v(S_y^2), v(S_z^2))$ leads to a contradiction.

In their original argument, Kochen and Specker used 117 such sets, but that number was reduced to 33 by Asher Peres.

THEOREM 3 (Non-existence of pre-existing values for positions and momenta. Clifton's theorem):

Let $\Psi(x_1, x_2) \in L^2(\mathbb{R}^2)$ be a function of two real variables (=the wave function of one particle in two dimensions or two particles on a line).

The position operators Q_1, Q_2 act as multiplication on functions:

$$Q_j \Psi(x_1, x_2) = x_j \Psi(x_1, x_2) \, , \quad j = 1, 2 \, ,$$

and the momentum operators P_1, P_2 act by differentiation:

$$P_j \Psi(x_1, x_2) = -i \frac{\partial}{\partial x_j} \Psi(x_1, x_2) \, , \quad j = 1, 2 \, .$$

Consider the set of analytic functions of one of the operators Q_1, Q_2, P_1, P_2. And let $\mathcal{B}$ be the set of products of such functions defining a self adjoint operator. Then, there does not exist a map

$$v : \mathcal{B} \to \mathbb{R}$$

such that:

1) $\forall A \in \mathcal{B}$ and for any real valued function f of a real variable,

$$v(f(A)) = f(v(A)),$$

2) $\forall A, B \in \mathcal{B}$ with $[A, B] = AB - BA = 0$:

$$v(AB) = v(A)v(B),$$

So, measurements of momentum must also be contextual (since position measurements do measure the real position).

THEOREM 4 (the von Neumann one, which is historically the first no hidden variables theorem).

$\nexists$ MAP $\quad v : \mathcal{A} \to \mathbb{R}$

$\mathcal{A}$= set of matrices or of operators, for example spin matrices and their products,

such that $\quad \forall\, A, B \in \mathcal{A}$

1) $v(A) \in$ {eigenvalues of A}

2) $v(A + B) = v(A) + v(B)$

even if A and B do not commute!.

So, same hypothesis as the Kochen-Specker theorem, but without the crucial provision:

$$[A, B] = 0.$$

PROOF (almost trivial)

Consider the Pauli matrices $A = \sigma_x$, $B = \sigma_y$, corresponding to a measurement of the spin along the x and y axes, respectively.

Then $\frac{\sigma_x+\sigma_y}{\sqrt{2}}$ corresponds to measuring the spin at an angle of $45°$ between the x and y axes.

All those matrices have eigenvalues equal to ± 1. Thus $v(A) = v(B) = \pm 1$, but also

$$v(\frac{\sigma_x + \sigma_y}{\sqrt{2}}) = \pm 1.$$

So we have $v(A + B) = \pm\sqrt{2}$, but $v(A) + v(B) = \pm 2$ or 0. Thus $v(A+B) = v(A)+v(B)$ does not hold.

End of proof!

The reason why von Neumann postulated $v(A+B) = v(A) + v(B)$ is probably because it holds when we average over the hidden variables: assuming hidden variables, for a given quantum state, means that the values of those variables vary between different experiments and that the quantum state determines the probability distribution of those variables. If we average over those variables and if the average agrees with the quantum mechanical prediction, we have

$$\mathbb{E}(v(A)) + \mathbb{E}(v(B)) = \mathbb{E}(v(A) + v(B)) \ ,$$

where $\mathbb{E}$ denotes the average.

$$\mathbb{E}(v(A)) + \mathbb{E}(v(B)) = \mathbb{E}(v(A) + v(B)) \ ,$$

holds because the average value of measurements of any physical quantity represented by a matrix or operator A when the quantum state is ψ, is given by $\langle\psi|A|\psi\rangle$, and that quantity satisfies $\langle\psi|A + B|\psi\rangle = \langle\psi|A|\psi\rangle + \langle\psi|B|\psi\rangle$.

But saying that, if an identity holds on average, then we may assume that it holds for every value over which the average is taken, is like saying that, if a function f satisfies

$$\int_{\mathbb{R}} f(x) d\mu(x) = 0,$$

with a certain probability measure μ, then we may assume that $f(x) = 0$ for all x. Hardly a natural assumption for a mathematician to make!

Moreover, Bell constructed a simple example of hidden "spin variables" that reproduces the quantum mechanical results *for a single spin*.

However, von Neumann was not modest concerning the significance of his theorem; after stating it, he concluded:

> It is therefore not, as is often assumed, a question of a reinterpretation of quantum mechanics — the present system of quantum mechanics would have to be objectively false, in order that another description of the elementary processes than the statistical one be possible.
>
> John von Neumann

Similar conclusions were drawn by Max Born, Wolfgang Pauli and others.

On the other hand, John Bell had a very critical attitude

> Then in 1932 [mathematician] John von Neumann gave a "rigorous" mathematical proof stating that you couldn't find a non- statistical theory that would give the same predictions as quantum mechanics. That von Neumann proof in itself is one that must someday be the subject of a Ph.D. thesis for a history student. Its reception was quite remarkable. The literature is full of respectful references to "the brilliant proof of von Neumann"; but I do not believe it could have been read at that time by more than two or three people.

Omni: Why is that?

Bell: The physicists didn't want to be bothered with the idea that maybe quantum theory is only provisional. A horn of plenty had been spilled before them, and every physicist could find something to apply quantum mechanics to. They were pleased to think that this great mathematician had shown it was so.

Yet the von Neumann proof, if you actually come to grips with it, falls apart in your hands! There is *nothing* to it. It's not just flawed, it's *silly*. If you look at the assumptions made, it does not hold up for a moment. It's the work of a mathematician, and he makes assumptions that have a mathematical symmetry to them. When you translate them into terms of physical disposition, they're nonsense. You may quote me on that: The proof of von Neumann is not merely false but *foolish*.

John Bell

Lesson: be cautious when trying to see the physical implications of mathematical theorems!

> The intellectual attractiveness of a mathematical argument, as well as the considerable mental labor involved in following it, makes mathematics a powerful tool of intellectual prestidigitation — a glittering deception in which some are entrapped, and some, alas, entrappers.
>
> Jacob Schwartz

Bell adds:

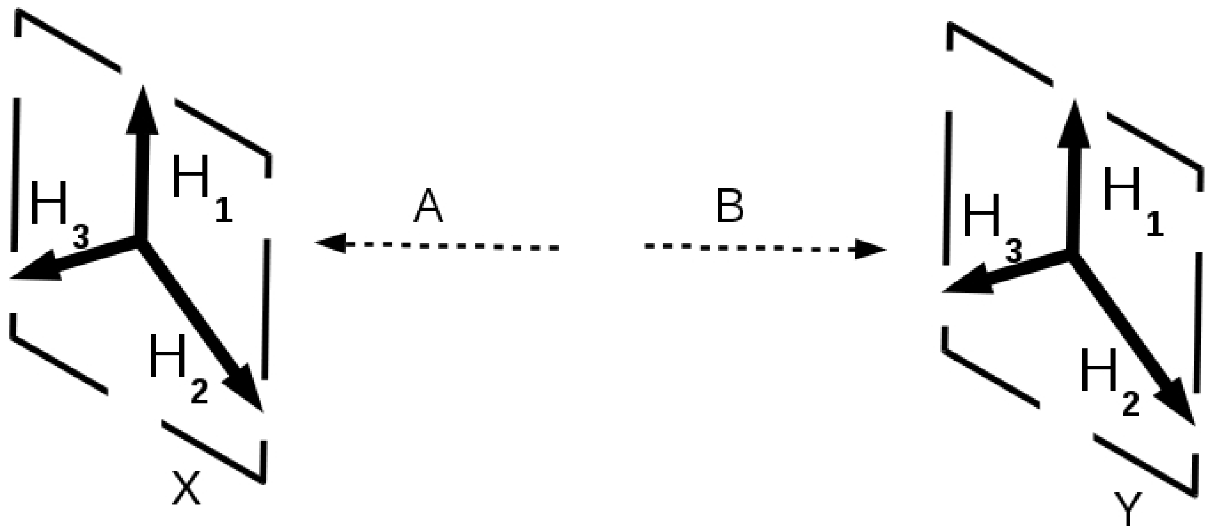
> But in 1952 I saw the impossible done. It was in papers by David Bohm. Bohm showed explicitly how parameters could indeed be introduced, into nonrelativistic wave mechanics, with the help of which the indeterministic description could be transformed into a deterministic one. More importantly, in my opinion, the subjectivity of the orthodox version, the necessary reference to the 'observer', could be eliminated.
>
> [. . .]

But why then had Born not told me of this ‘pilot wave’? If only to point out what was wrong with it? Why did von Neumann not consider it? More extraordinarily, why did people go on producing ‘impossibility’ proofs, after 1952, and as recently as 1978?

John Bell

THEOREM 5 (Bell's part of the nonlocality proof).

REMINDER: Bell's theorem is also a no hidden variable theorem (but remember that, combined with the EPR argument, it is a nonlocality proof).



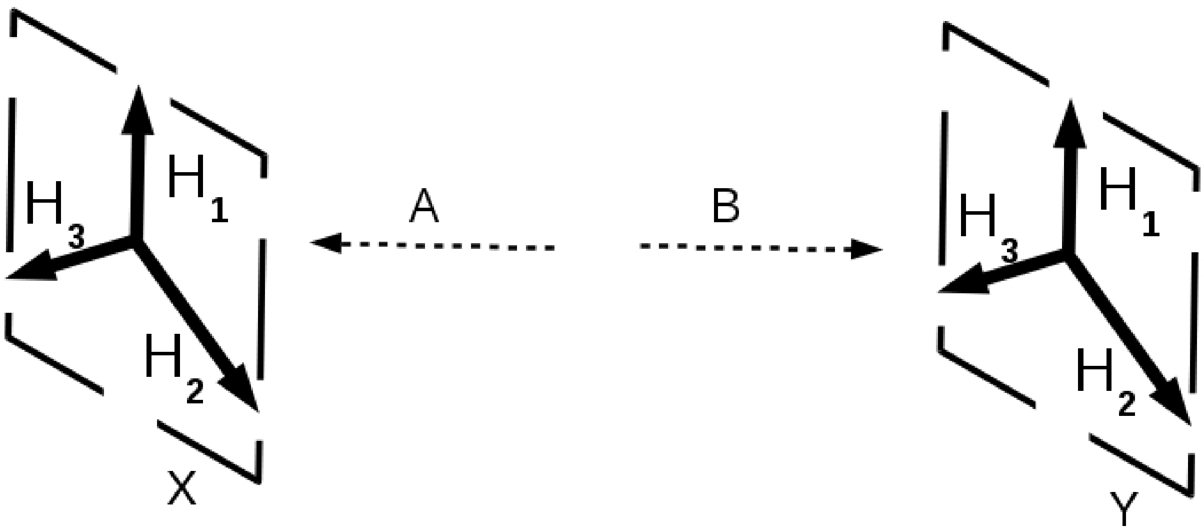
Consider three possible orientations for the magnetic field, denoted H_1, H_2, H_3, in a plane perpendicular to the motion of the particles.

When the orientations are the same on both sides, the two particles always go in opposite directions.

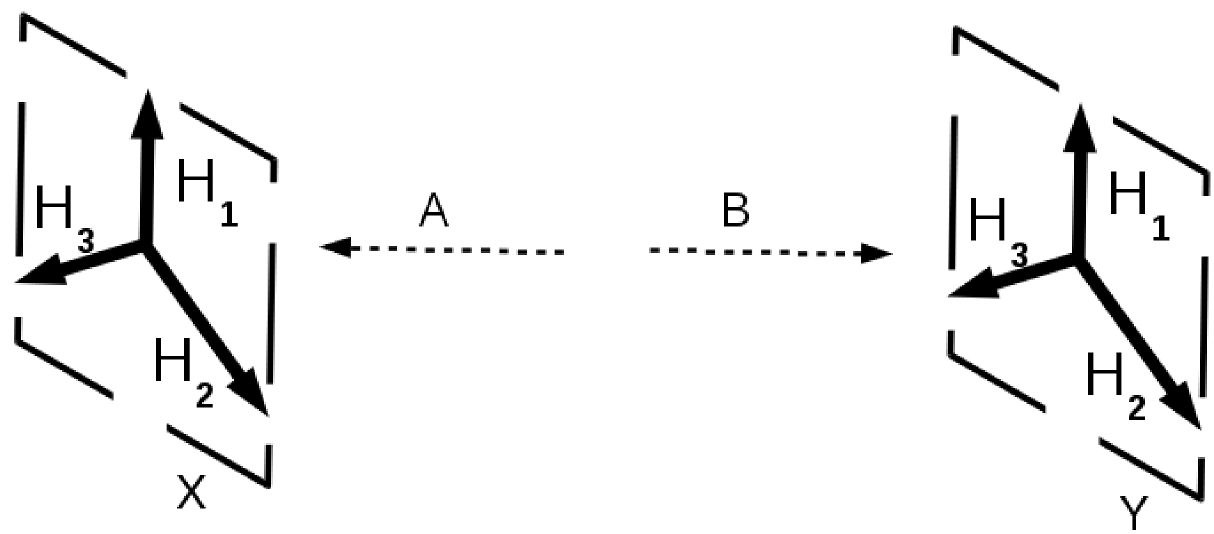
This holds for the state:

$$| \text{ state of the two particles } >$$

$$= \frac{1}{\sqrt{2}}(|A\ 1 \uparrow> |B\ 1 \downarrow> -|A\ 1 \downarrow> |B\ 1 \uparrow>)$$
$$= \frac{1}{\sqrt{2}}(|A\ 2 \uparrow> |B\ 2 \downarrow> -|A\ 2 \downarrow> |B\ 2 \uparrow>)$$
$$= \frac{1}{\sqrt{2}}(|A\ 3 \uparrow> |B\ 3 \downarrow> -|A\ 3 \downarrow> |B\ 3 \uparrow>)$$



So, introduce "random variables" $A(\alpha) = \pm 1$, $B(\alpha) = \pm 1$, for $\alpha = 1, 2, 3$ labelling the direction, and where $A(\alpha) = +1$ means that the A particle will go in the direction of the field when the latter is oriented in direction α, and $A(\alpha) = -1$ means that the A particle will go in the direction opposite to the one of the field, and similarly for $B(\alpha) = \pm 1$.



The perfect anti-correlations imply:

$$A(\alpha) = -B(\alpha)$$

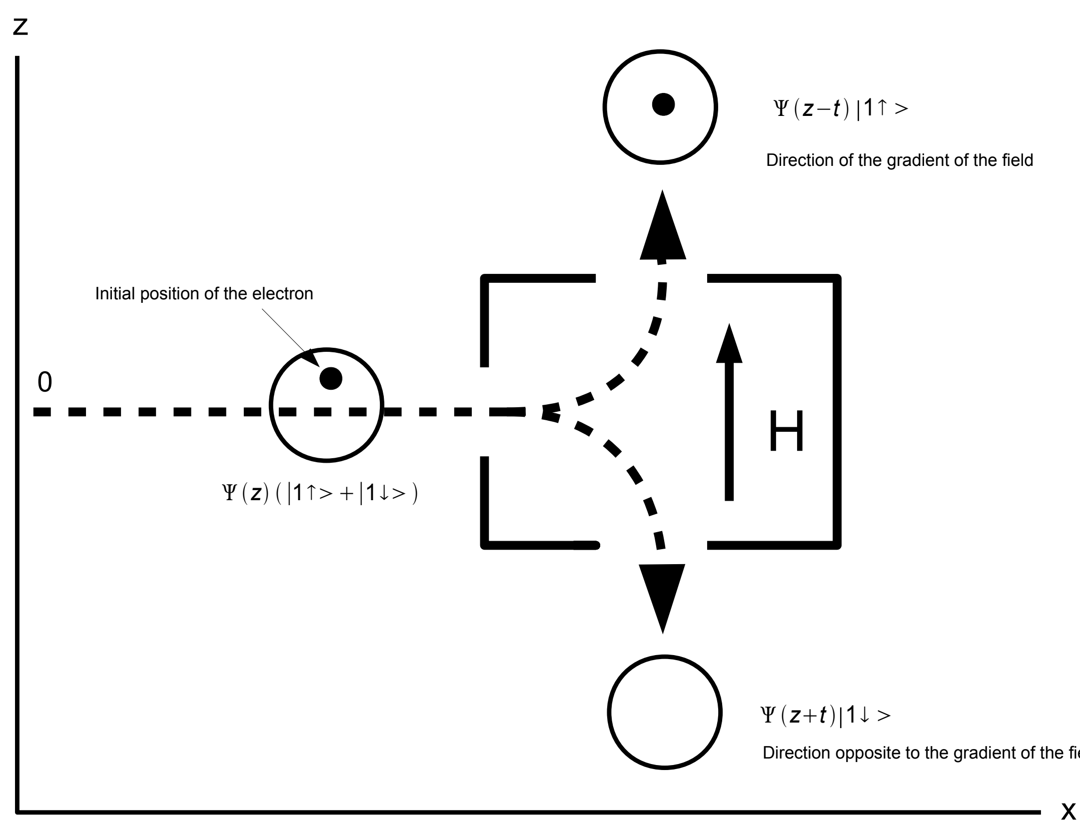
$\forall \alpha = 1, 2, 3.$

These random variables are "hidden variables" and Bell showed that simply assuming that they exist leads to a contradiction.

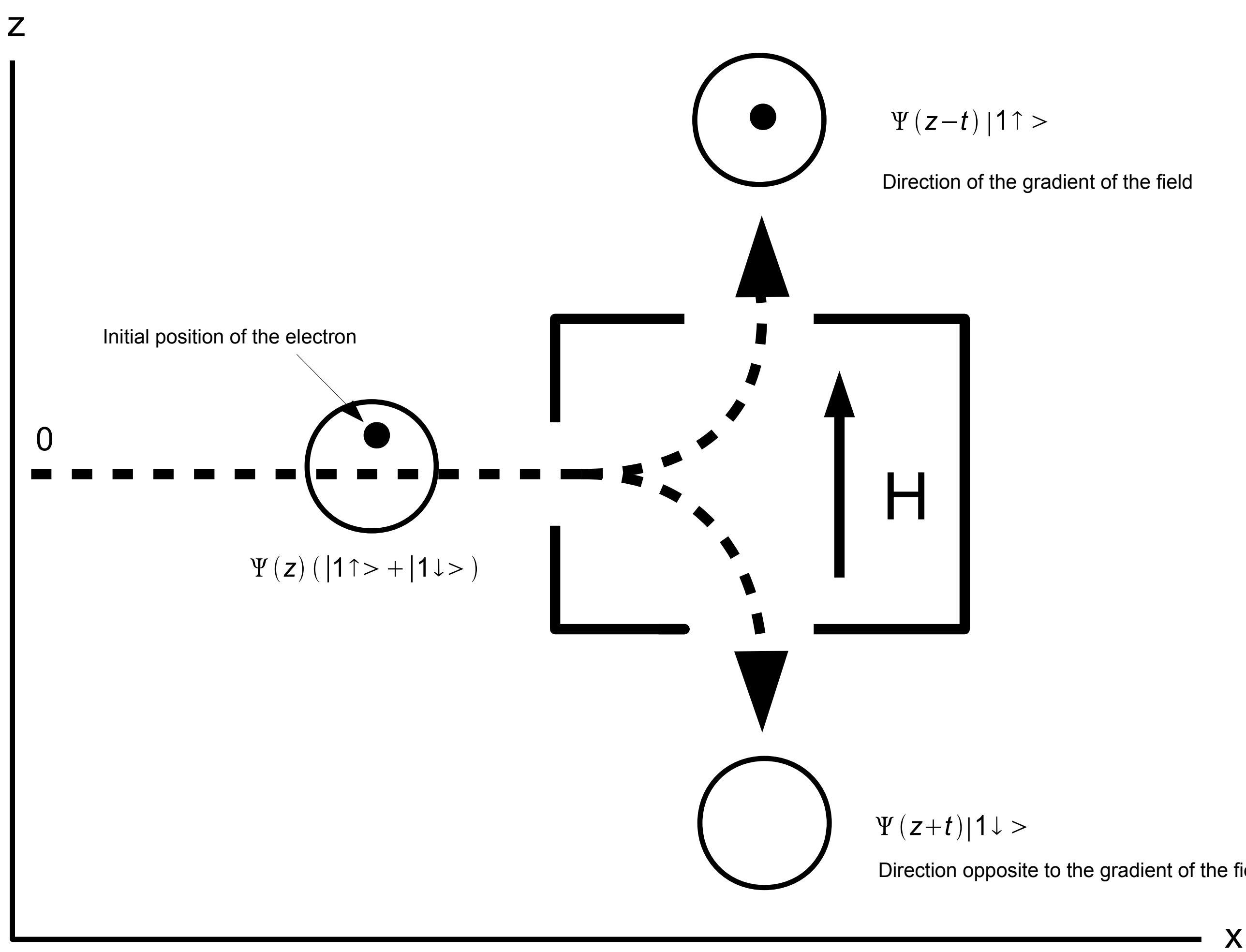
ANOTHER REMINDER: HOW DOES THE DE BROGLIE-BOHM THEORY AVOID BEING REFUTED BY THE NO HIDDEN VARIABLES THEOREMS?

BY NOT INTRODUCING “HIDDEN VARIABLES” FOR QUANTITIES OTHER THAN POSITIONS!

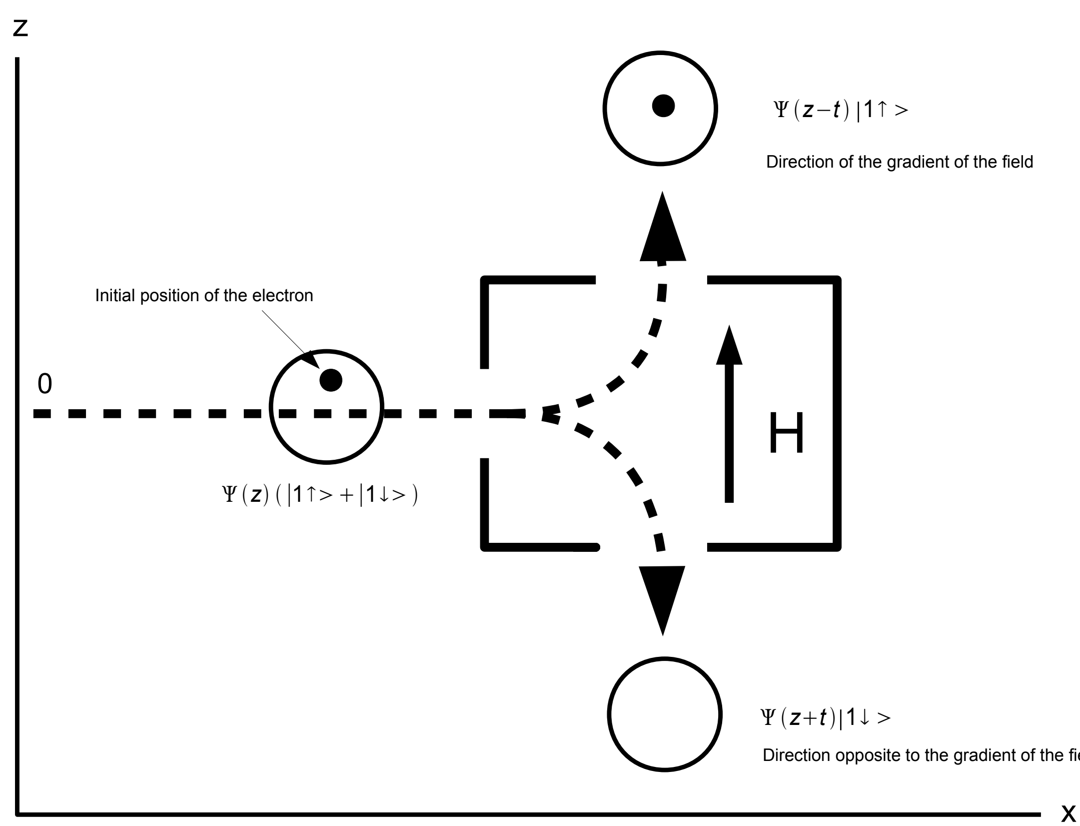
Consider a Stern-Gerlach apparatus "measuring" the spin. Let H be the magnetic field. The arrow in the picture indicates the direction of the gradient of that field.



The $|1\uparrow>$ part of the state always goes in the direction of the gradient of the field, and the $|1\downarrow>$ part always goes in the opposite direction.



But if the particle is initially in the upper part of the support of the wave function (for a symmetric wave function), it will always go upwards. That is because there is a nodal line in the middle of the figure that the particles cannot cross.



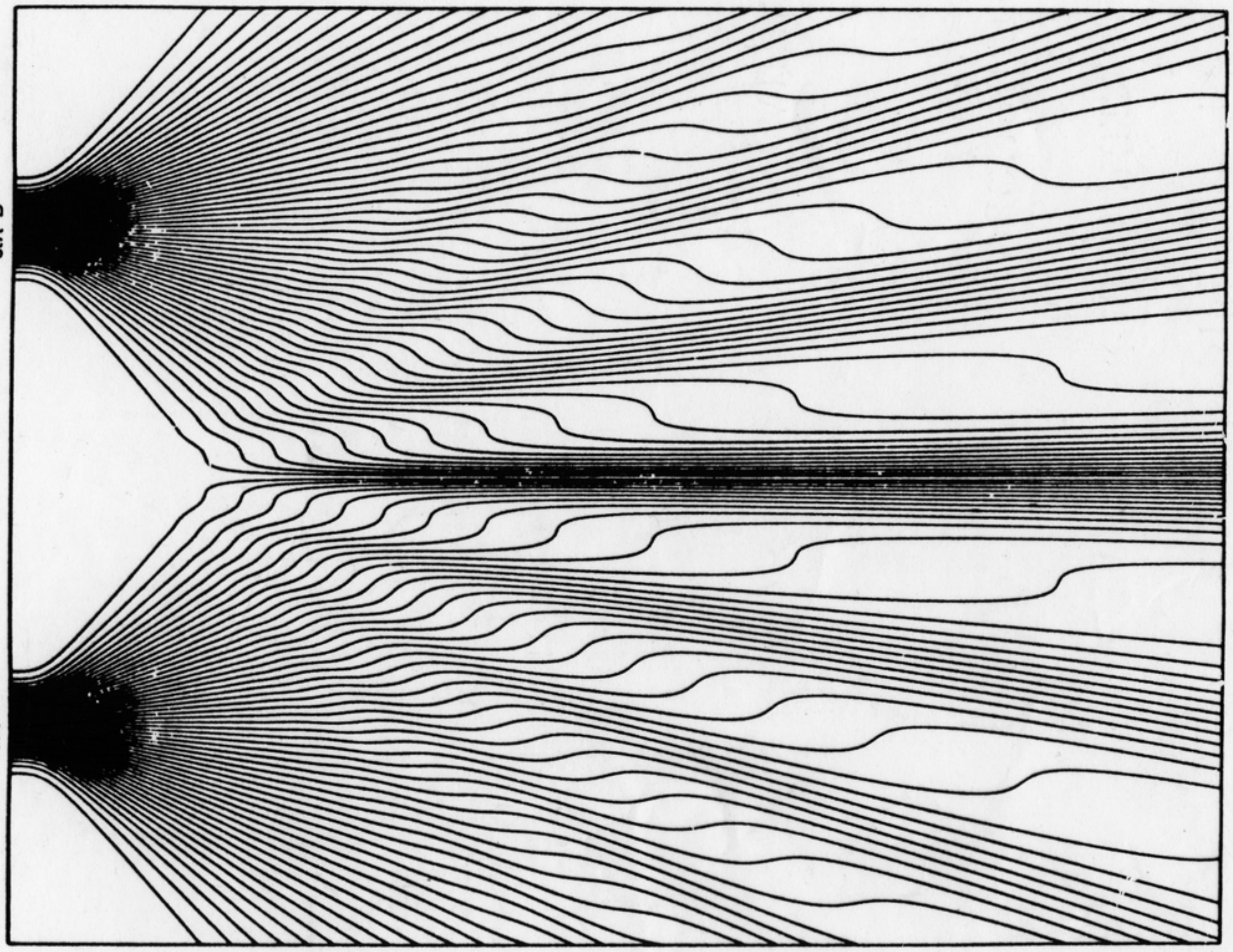
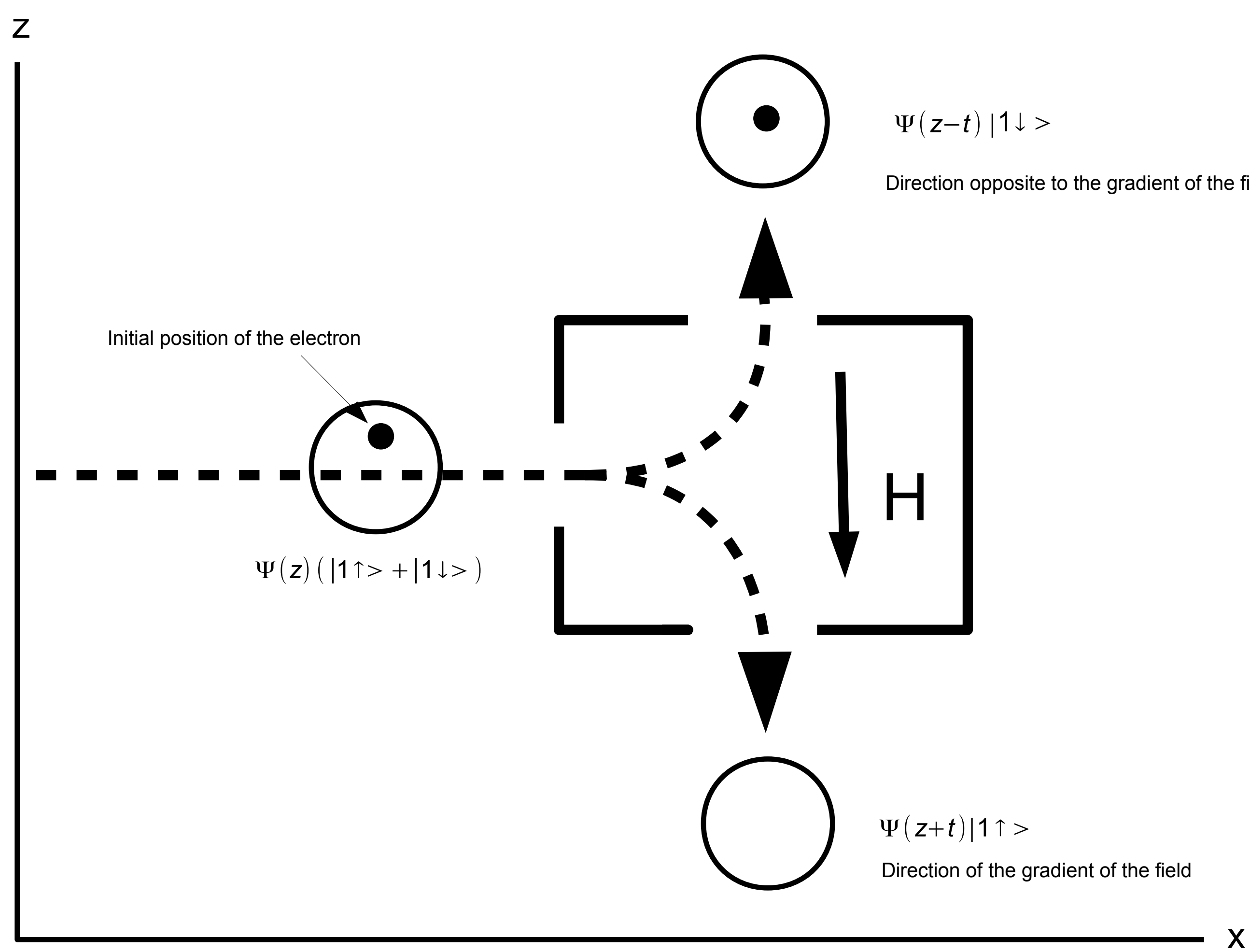


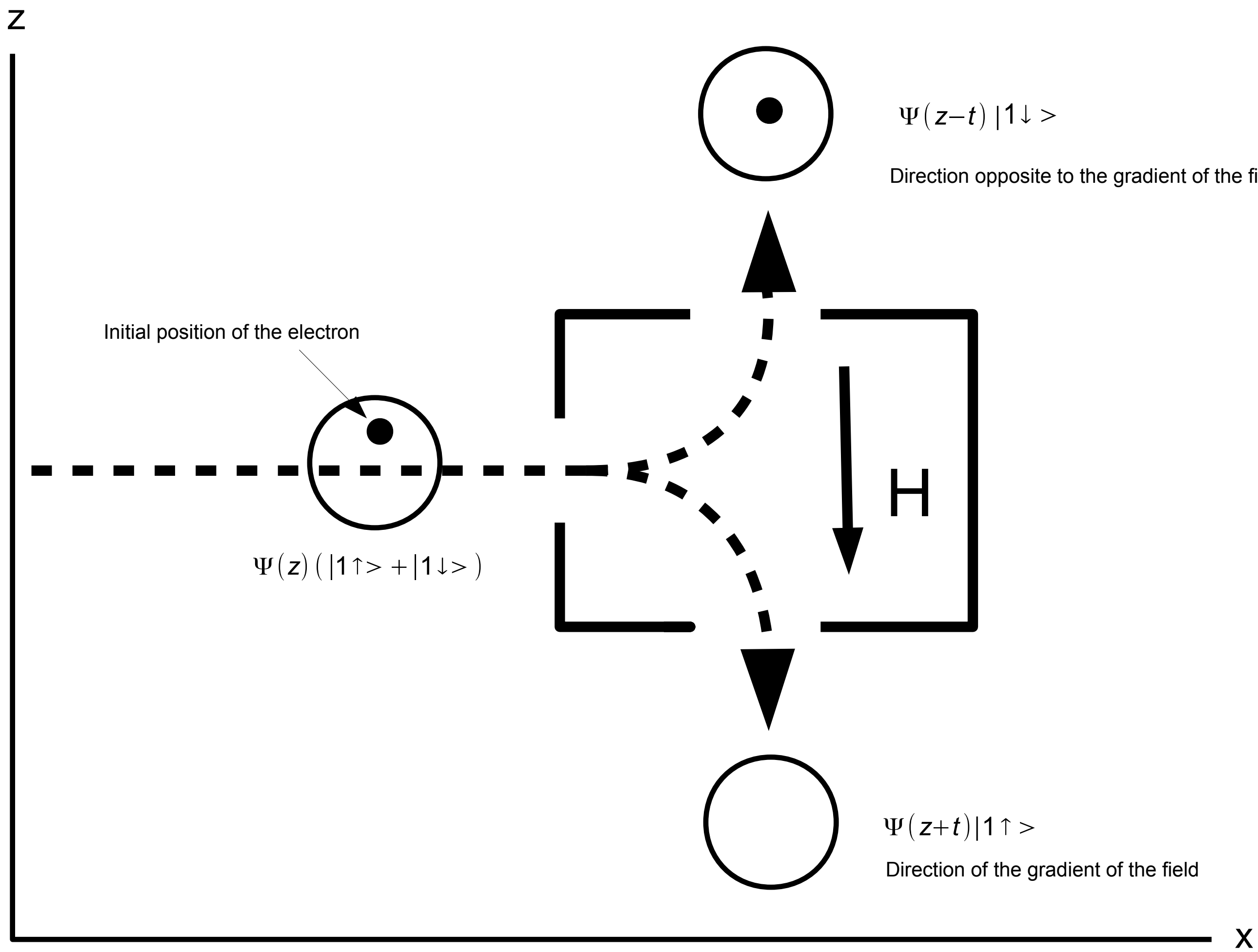
Figure 9.3 Trajectories for two Gaussian slits with a Gaussian distribution of initial positions at the slits.

as here

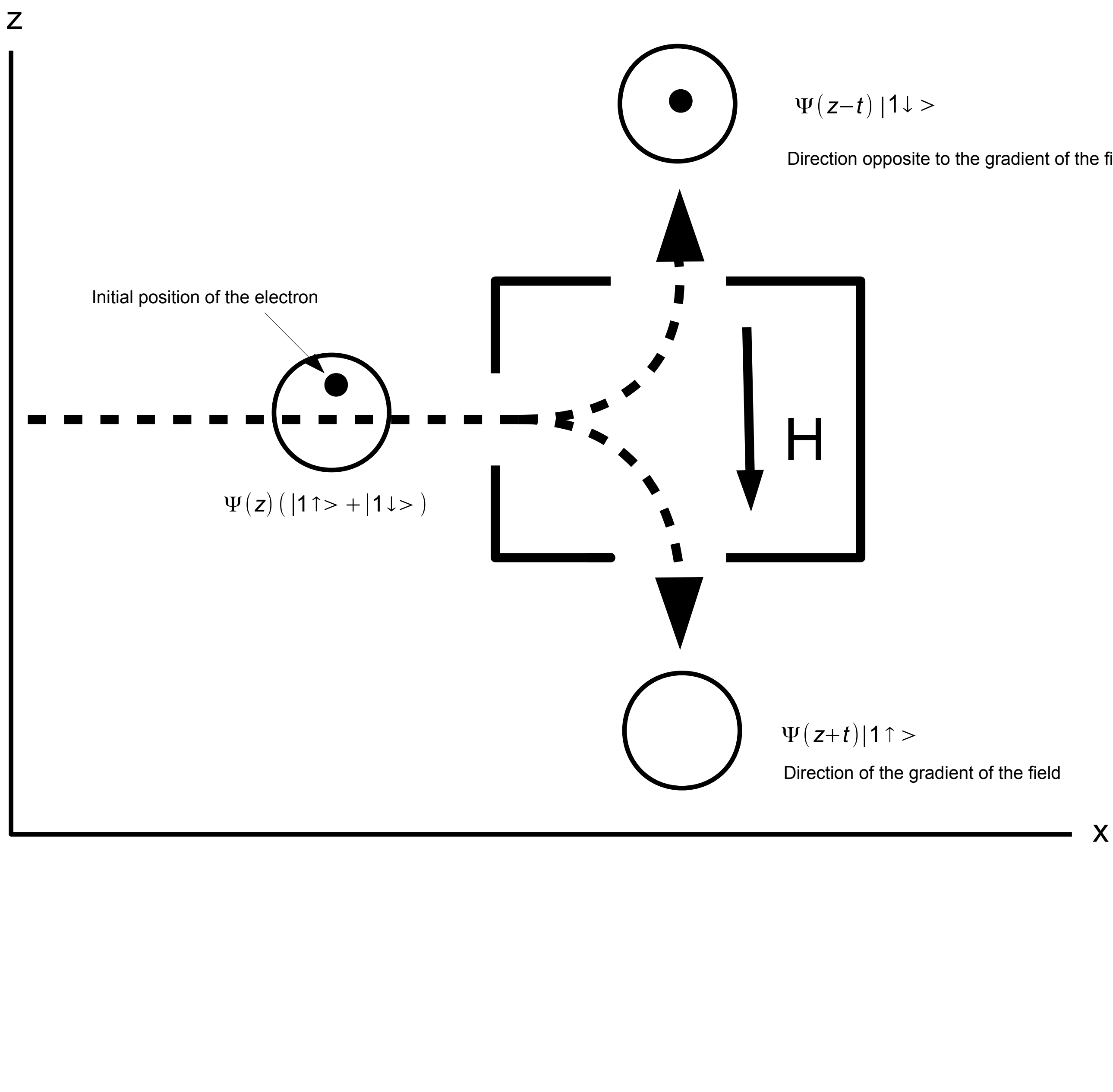
Now, repeat the same experiment, but with the direction of the gradient of the field reversed, and let us assume that the particle starts with *exactly the same wave function and the same position as before.*



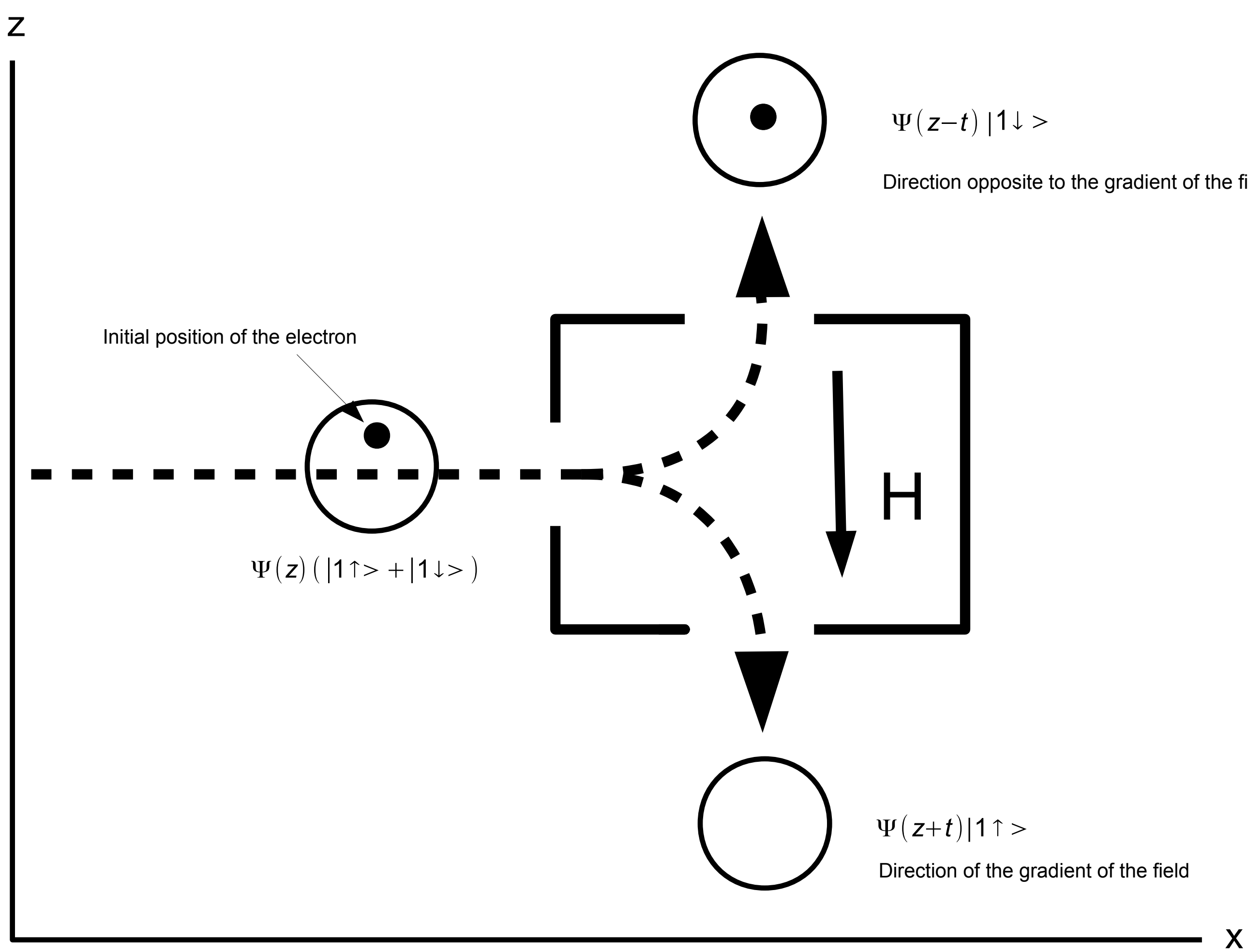
The particle is initially in the upper part of the support of the wave function, and, thus, it will still go upwards, because of the nodal line.



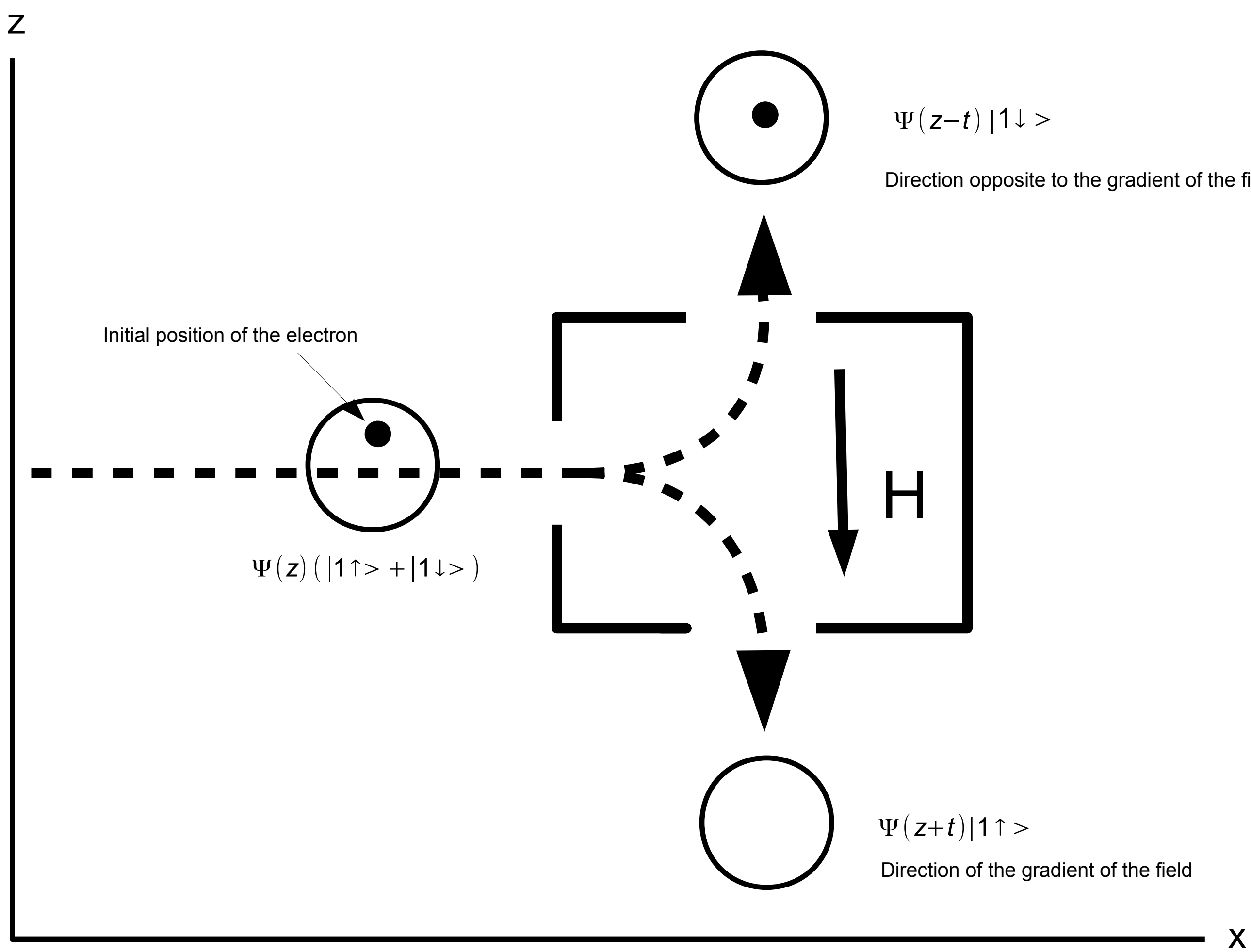
But going upwards means now going in the direction *opposite* to the one of the gradient of the field (since the latter is reversed).



So, the particle whose spin was "up", will "have" its spin "down", although one "measures" exactly the same quantity (the spin in the vertical direction), with *exactly the same initial conditions (for both the wave function and the position of the particle).*



So, the particle whose spin was "up", will "have" its spin "down", although one "measures" exactly the same quantity, with exactly the same initial conditions (for both the wave function and the position of the particle), but with two different arrangements of the apparatus.



SOME REMARKS ON "QUANTUM LOGIC"

An idea that has had some popularity, partly due to the (deserved) reputation of its founders: Birkhoff and von Neumann.

If p and q are propositions, then the elementary calculus of propositions is based on the following operations:

(1) $p \vee q$: p or q.

(2) $p \wedge q$: p and q.

(3) $\neg p$: the negation of p.

Similar operations can be done with subsets, A, B, $\ldots$ of a set E:

(1) $p \vee q$: $\to A \cup B$.

(2) $p \wedge q$: $\to A \cap B$.

(3) $\neg p$:$\to A^c = E \setminus A$.

Think of A as the set of elements for which proposition p holds and B as the set of elements for which proposition q holds.

These algebraic structures, with their obvious properties, in particular distributiveness:

(1) $p \wedge (q \vee q') = (p \wedge q) \vee (p \wedge q')$.

(2) $p \vee (q \wedge q') = (p \vee q) \wedge (p \vee q')$.

form a **Boolean algebra**.

Now consider subspaces, $E, F, \ldots,$ of a Hilbert space $\mathcal{H}$. We can do something similar:

(1) $p \vee q$: $\rightarrow$ $\overline{E \cup F}$. The smallest subspace containing of the union of subspaces E and F.

(2) $p \wedge q$: $\rightarrow E \cap F$.

(3) $\neg p$:$\rightarrow$ $E^{\perp}$. The orthocomplement of E in $\mathcal{H}$.

It is similar to the previous structure but without distributiveness: for vector spaces E, F, G: $E \cap (\overline{F \cup G}) \neq \overline{\{E \cap F) \cup (E \cap G})$.

Take for example F the x axis, G the y axis in $\mathbb{R}^2$ and E the line $y = x$.

Then $\overline{F \cup G} = \mathbb{R}^2$, $E \cap (\overline{F \cup G})$ is the line $y = x$ but $E \cap F$ and $E \cap G$ are equal to the point 0.

As far as mathematics is concerned, this is OK and maybe “nice”.

But there is a tendency to associate subspaces of $\mathcal{H}$ with “propositions”. If Ψ is a quantum state, A an operator and E a subspace of eigenvector(s) of A, one would ask:

Does the system whose quantum state is Ψ have the properties associated with the eigenvalue(s) associated to E ?

The answer to that question would be given by a “measurement” of operator A or of the projector P_E on the subspace E.

The answer would be “yes” or “no”.

Hence the idea that quantum mechanics is just a series of “yes-no” questions.

The next step is to wonder what to do with the non-Boolean (non distributive) structure of the "quantum propositions", i.e. of subspaces of $\mathcal{H}$.

Again, if it is a mere "way to speak" or analogy, it is OK.

But if one takes that language literally, one is again guilty of NRAO!

The no hidden variables theorems show that subspaces of $\mathcal{H}$ in which the quantum state could "collapse" after a measurement do not correspond to pre-existing properties of the system.

This is clearly seen in de de Broglie-Bohm. theory.

The worst step is to decide that we need a new "quantum logic". This was suggested seriously by some physicists and also by philosophers (including logicians).

For example, Quine, one of the most famous analytical philosophers of the second half of the twentieth century wrote:

> Revision even of the logical law of the excluded middle has been proposed as a means of simplifying quantum mechanics; and what difference is there in principle between such a shift and the shift whereby Kepler superseded Ptolemy, or Einstein Newton, or Darwin Aristotle?
>
> Willard Van Orman Quine

This was put forward even more explicitly by Hilary Putnam (at some point).

This is a huge mistake, because ordinary logic is the prerequisite of all thinking, in particular of thinking about the natural world.

Saying that we should modify logic in order to understand the natural world amounts to saying that we must renounce understanding that world.

As the philosopher of science Tim Maudlin once nicely said:

"There is no point in arguing with somebody who does not believe in logic."

Again, what John Bell said was exactly to the point:

> When it is said that something is 'measured' it is difficult not to think of the result as referring to some pre-existing property of the object in question. This is to disregard Bohr's insistence that in quantum phenomena the apparatus as well as the system is essentially involved. [...] When one forgets the role of the apparatus, as the word 'measurement' makes all too likely, one despairs of ordinary logic - hence 'quantum logic'. When one remembers the role of the apparatus, ordinary logic is just fine.
>
> John Bell

There is another rather fashionable approach to quantum mechanics that runs into trouble because of the no hidden variables theorems:

The decoherent or consistent histories approach to quantum mechanics (Gell-Mann, Hartle, Griffiths, Omnès).

The basic idea is to assign probabilities to various "histories" or sequence of events that are supposed to be real events happening in the world and not just "results of measurements".

But it assumes "operator democracy" namely the idea that all operators should be treated in the same way (you can already see that this will be the source of the troubles, if you think of the no hidden variables theorems).

A “history” could be something like : “the spin of the particle is oriented in this direction at time t_1 and in that direction at time t_2 etc.”

Or “the particle goes trough the upper slit and ends up at point x on the second wall”.

The probability of such histories is of course the same as the one one would have if these sequence of events were replaced by sequences of measurements.

But one has to be careful, because of interference.

For example, consider the following history for a spin, with stationary dynamics: at time t_0, we have $\sigma_z = +1$, at time t_1, $\sigma_x = 1$, and at time t_2, $\sigma_z = -1$.

This has probability $\frac{1}{4}$ (exercise: check).

But we have the same probability for the following history: at time t_0, we have $\sigma_z = +1$, at time t_1, $\sigma_x = -1$, and at time t_2, $\sigma_z = -1$.

But if we sum over the events "at time t_1, $\sigma_x = 1$" and "at time t_1, $\sigma_x = -1$" (these are the only two possibilities), we get the history: "at time t_0, $\sigma_z = +1$, and at time t_2, $\sigma_z = -1$", which has probability 0!

Or consider the following pair of "histories": "the particle goes through the upper slit and is detected at a certain place x on the screen", and "the particle goes through the lower slit and is detected at x on the screen".

The probability of being detected at x should be the sum of the probabilities of those two sequences of events, but the probability of being detected at x when both slits are open is different from the sum of the probabilities of being detected at x and going through each slit, when only one slit is open. This means that *a priori* there is a difficulty in assigning probabilities consistently to those histories.

Of course, if one considers two histories made of **different** sequences of events: "the particle is detected at the upper slit and is detected again at x on the screen", and "the particle is detected at the lower slit and is detected again at x on the screen", then one can assign probabilities consistently to such a pair of histories, because the detection at the slits in effect collapses the wave function and thus destroys the interference phenomenon.

That is why Gell-Mann, Hartle, Griffiths, Omnès introduce conditions of decoherence between members of a family of histories in such a way as to allow the application of the usual rules of probability to that family.

We will not discuss in detail those conditions, because what they try to avoid are inconsistencies that might arise when one attributes probabilities to events occurring at different times (as in the example of the particle going through the upper or lower slit and being detected at a certain place x on the screen) and the problem that we want to focus on, in the decoherent histories approach, occurs already when one considers events happening at a single time.

Indeed, and this cannot be stressed enough, the originality and the merits of the decoherent histories approach are that it tries to make sense of probabilities of real events happening in the world, whether we observe them or not. If it did not try to do that, then there would be nothing new, since the formulas for sequences of events occurring at different times in that approach are just the usual quantum mechanical ones, obtained from the combination of Schrödinger evolution and the collapse rule.

Robert Griffiths is very explicit about the decoherent histories goal:

> "For example, one can show that a properly constructed measuring apparatus will reveal a property that the measured system had before the measurement, and might well have lost during the measurement process. The probabilities calculated for measurement outcomes (pointer positions) are identical to those obtained by the usual rules found in textbooks. What is different is that by employing suitable families of histories one can show that measurements actually measure something that is there, rather than producing a mysterious collapse of a wave function."

However, by considering as real events the values taken by arbitrary observables, this leads to a problem, raised in papers by Sheldon Goldstein and others, a problem which exists even if one considers "histories" occurring *at a single time* and for which the decoherence condition is trivially satisfied.

Using four decoherent histories, based on an example due to Lucien Hardy, Sheldon Goldstein is able to show that the decoherent histories approach runs into a contradiction.

Consider four quantities A, B, C, D, associated with a pair of spin 1/2 particles (defined in the Appendix and that you may analyze as an exercice), which have the following properties:

(1) A and C can be measured simultaneously (this means that the matrices representing those quantities commute) and if one gets $A = 1$, then $C = 1$.

(2) B and D can be measured simultaneously, and if one gets $B = 1$, then $D = 1$.

(3) C and D can be measured simultaneously, but it never happens that both $C = 1$ and $D = 1$.

(4) A and B can be measured simultaneously, and it sometimes happens that both $A = 1$ and $B = 1$.

(1) A and C can be measured simultaneously and if one gets $A = 1$, then $C = 1$.

(2) B and D can be measured simultaneously, and if one gets $B = 1$, then $D = 1$.

(3) C and D can be measured simultaneously, but it never happens that both $C = 1$ and $D = 1$.

(4) A and B can be measured simultaneously, and it sometimes happens that both $A = 1$ and $B = 1$.

Each of these four statements corresponds to a decoherent history in a trivial way, since both quantities, in each of the pairs involved here, can be measured simultaneously, and we are considering everything at a single time.

So one can assign probabilities to those events in a consistent way. For example, since "A and C can be measured simultaneously and if one gets $A = 1$, then $C = 1$", the probability that $C = 1$, given that $A = 1$, is equal to 1.

(1) A and C can be measured simultaneously and if one gets $A = 1$, then $C = 1$.

(2) B and D can be measured simultaneously, and if one gets $B = 1$, then $D = 1$.

(3) C and D can be measured simultaneously, but it never happens that both $C = 1$ and $D = 1$.

(4) A and B can be measured simultaneously, and it sometimes happens that both $A = 1$ and $B = 1$.

But these four statements cannot all be true, because when it happens that $A = 1$ and $B = 1$, as it sometimes does, by (4), one must have, by (1) and (2), that $C = 1$ and $D = 1$, which is impossible because of (3).

Of course, each of the statements above is true if they all refer to results of measurements. But then no contradiction arises, because the measurements to which they refer are different and cannot be performed simultaneously since they do perturb the system (as reflected by the collapse rule in ordinary quantum mechanics and understood through the analysis of measurements in the context of the de Broglie–Bohm theory).

The contradiction here is similar to what happens with the no hidden variables theorems: thinking that "measured values" correspond to real events in the world!

As stressed by Goldstein, the inconsistency is a problem, but only from the decoherent history point of view:

> It is important to appreciate that, for orthodox quantum theory (and, in fact, even for Bohmian mechanics), the four statements above, if used properly, are not inconsistent, because they then would refer merely to the outcomes of four different experiments, so that the probabilities would refer, in effect, to four different ensembles.

However, the whole point of decoherent histories is that such statements refer directly, not to what would happen were certain experimental procedures to be performed, but to the probabilities of occurrence of the histories themselves, regardless of whether any such experiments are performed.

Sheldon Goldstein

The response from Gell-Mann and Hartle and from Griffiths consists basically in saying that there is no decoherent history including the four operators A, B, C, and D. This is true, because A and D cannot be measured simultaneously, nor can B and C. But that answer misses the point, which is that each decoherent history is supposed to be a statement about real events happening in the world, to which truth values can be assigned. So each of the above statements is meant to be true, but they cannot all be true, since, taken together, they lead to a contradiction.

The next step taken in particular by Griffiths is similar to "quantum logic": redefine the rules through which truth values are assigned:

If a proposition p is true, and a proposition q is true, then (in ordinary logic) the proposition $p \wedge q$ is true.

The only way to maintain that each of the above propositions $A - D$ is true is to "ban" this rule since propositions $A - D$ taken together lead to a contradiction.

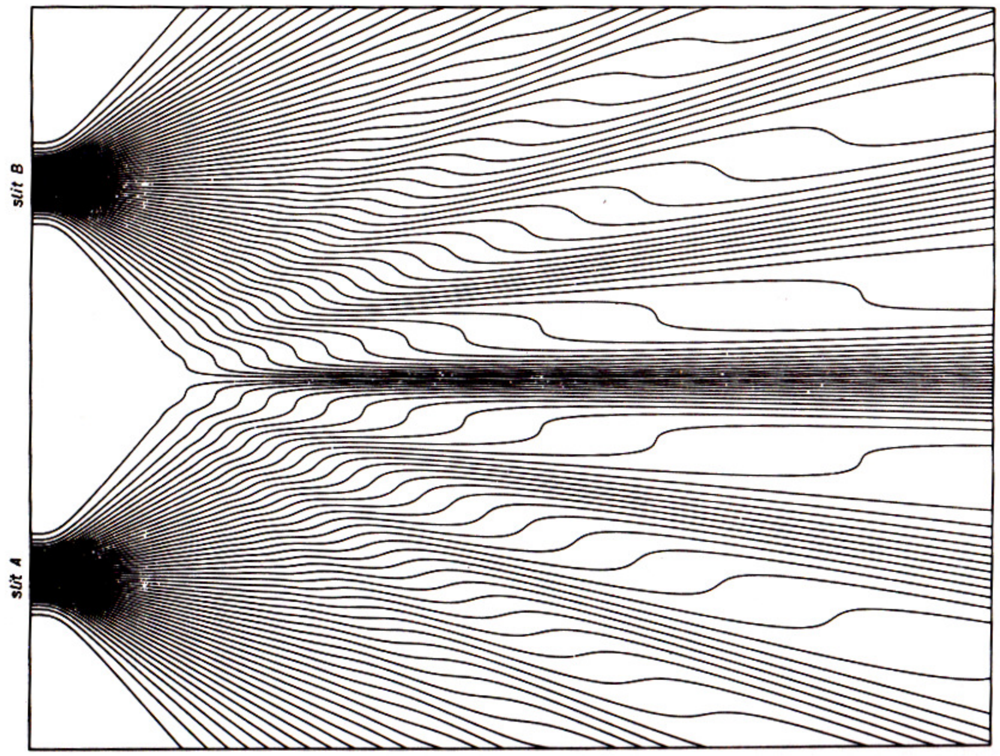
But that is SILLY!

The decoherent history approach commits two "sins" at once:

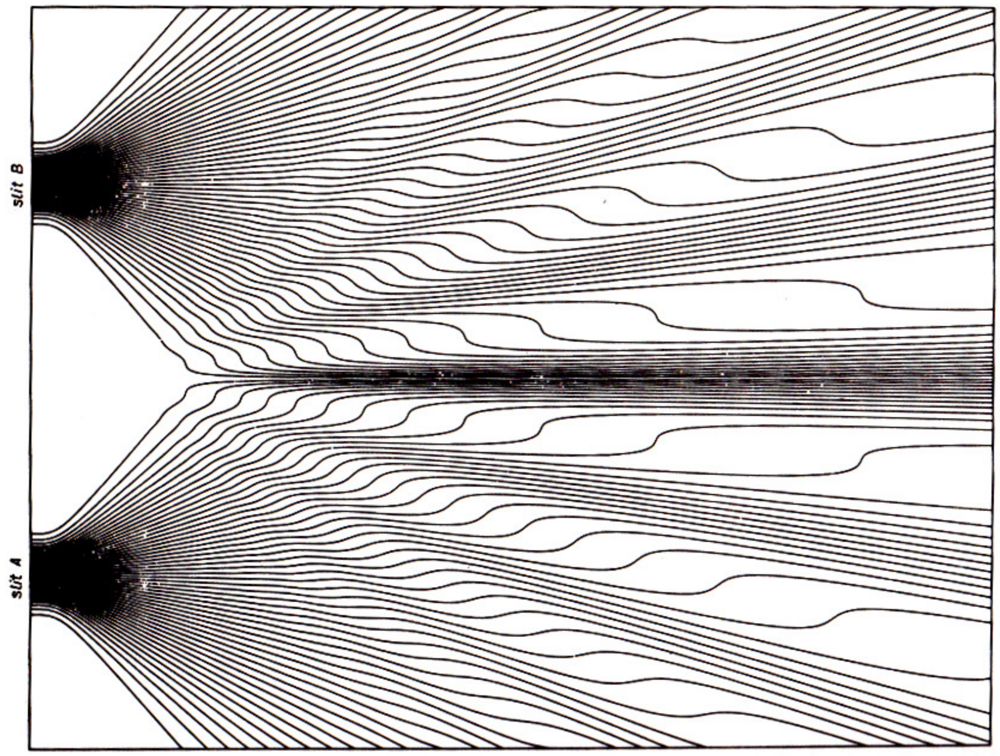
NRAO

Alternative logic.

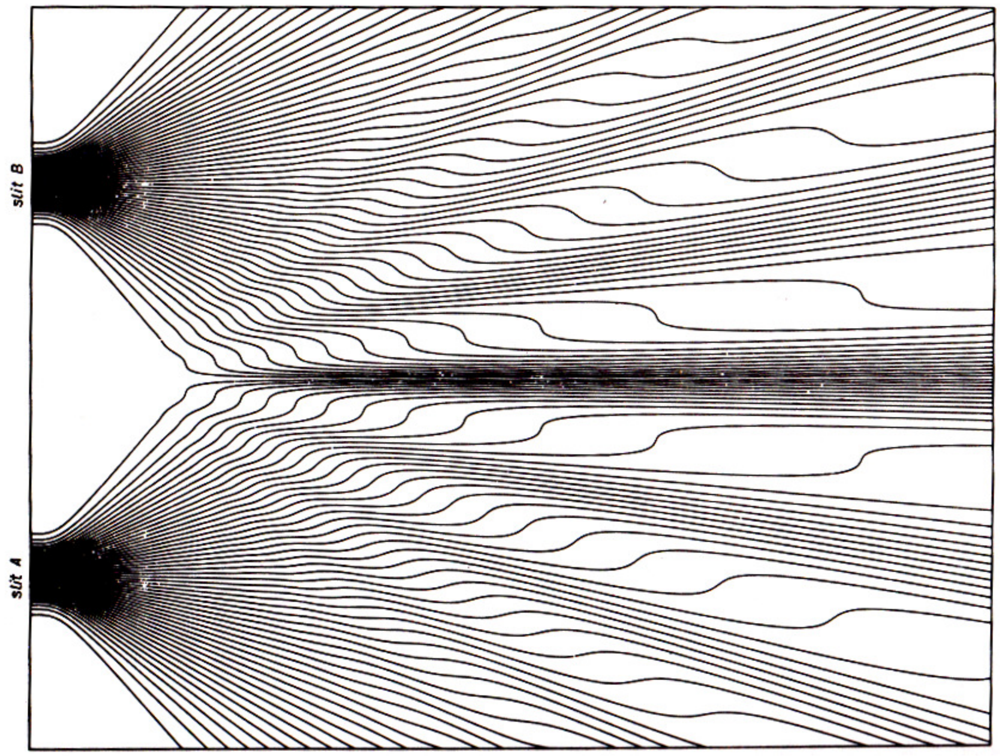
Finally, one may contrast the decoherent history approach and the de Broglie–Bohm theory. After all, by assigning probabilities, at each time, to particle positions, the de Broglie–Bohm theory also assigns probabilities to certain “histories”, namely the particle trajectories.



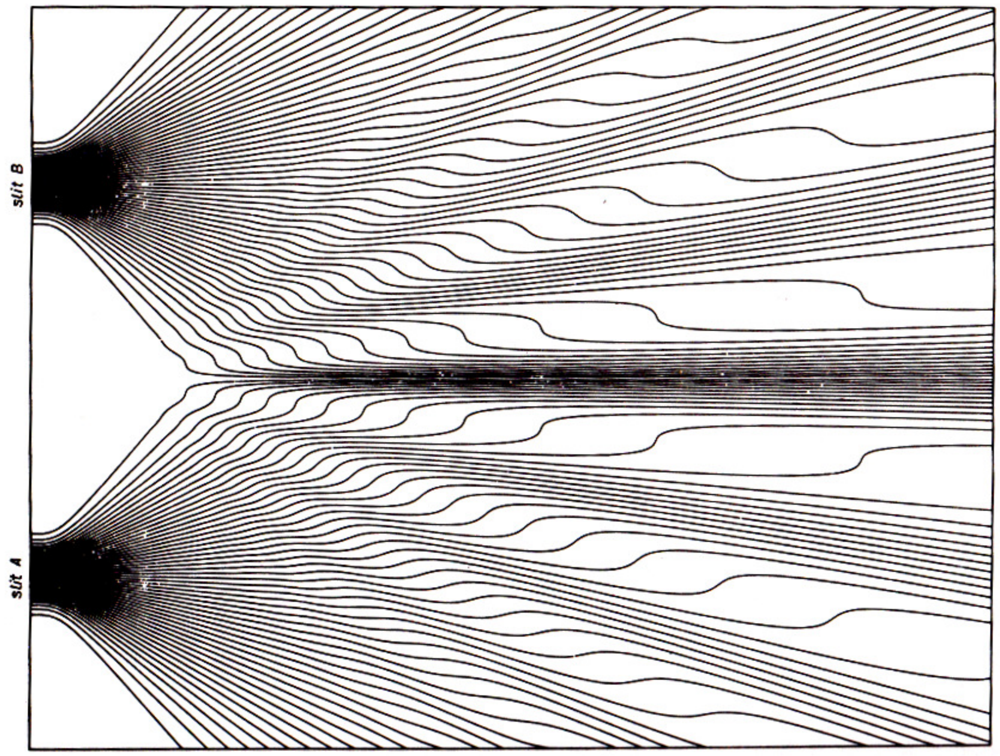
Indeed, they do assign probability densities to the history of a particle going through the upper slit and ending at a given point x on the screen *and* to the history of a particle going through the lower slit and ending at the same point x on the screen, but in a consistent way.



Looking at this figure, we see that, if the point x is in the upper half of the screen, then the probability of going through the lower slit and ending at x is zero, while the conditional probability of going through the upper slit, given that one ends at x, is one.



So, there is no inconsistency in these assignments of probabilities: the probability density of ending at x *is* (for this assignment of probabilities) the sum of the probability densities of going through the lower slit and ending at x (which is zero) and of of going through the upper slit and ending at x (which equals the probability density of ending at x).



This may sound counterintuitive, but this assignment of probabilities is consistent and its counterintuitive nature arises solely from a quantum mechanical “intuition” that is deeply linked with results of measurements.

Finally, let me mention that there exists a general notion of what a measurement is in quantum mechanics: a Positive Operator Valued Measure (POVM); see Tumulka's lectures, p. 99.

Lesson from this lecture: be realist, but beware of naive realism about operators!

And, when dealing with "fancy" mathematics, remember Jacob Schwartz' advice:

> The intellectual attractiveness of a mathematical argument, as well as the considerable mental labor involved in following it, makes mathematics a powerful tool of intellectual prestidigitation — a glittering deception in which some are entrapped, and some, alas, entrappers.
>
> Jacob Schwartz

APPENDIX 1 PROOF OF MERMIN'S NON HIDDEN VARIABLES THEOREM

We use the standard Pauli matrices σ_x , σ_y:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} ,$$

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} ,$$

We consider a couple of each of those matrices, σ_x^i, σ_y^i, $i = 1, 2$, where tensor products are implicit: $\sigma_x^1 \equiv \sigma_x^1 \otimes \mathbf{1}$, $\sigma_x^2 \equiv \mathbf{1} \otimes \sigma_x^2$, etc., with $\mathbf{1}$ the unit matrix. These operators act on $\mathbf{C}^4$.

The following identities are well known and easy to check:

i)

$$(\sigma_x^i)^2 = (\sigma_y^i)^2 = (\sigma_z^i)^2 = \mathbf{1} \ ,$$

for $i = 1, 2$.

ii) Different Pauli matrices anticommute:

$$\sigma_\alpha^i \sigma_\beta^i = -\sigma_\beta^i \sigma_\alpha^i \ ,$$

for $i = 1, 2$, and $\alpha, \beta = x, y, z$, $\alpha \neq \beta$.

iii) Pauli matrices associated with different variables commute:

$$\sigma_\alpha^1 \sigma_\beta^2 = \sigma_\beta^2 \sigma_\alpha^1 \ ,$$

where $\alpha, \beta = x, y, z$.

Consider now the identity

$$\sigma_x^1\sigma_y^2\sigma_y^1\sigma_x^2\sigma_x^1\sigma_x^2\sigma_y^1\sigma_y^2 = -\mathbf{1} \ ,$$

which follows, using first $\sigma_\alpha^i\sigma_\beta^i = -\sigma_\beta^i\sigma_\alpha^i$,

and $\sigma_\alpha^1\sigma_\beta^2 = \sigma_\beta^2\sigma_\alpha^1$,

to move σ_x^1 in the product from the first place (starting from the left) to the fourth place, a move that involves one anticommutation and two commutations , viz.,

$$\sigma_x^1\sigma_y^2\sigma_y^1\sigma_x^2\sigma_x^1\sigma_x^2\sigma_y^1\sigma_y^2 = -\sigma_y^2\sigma_y^1\sigma_x^2\sigma_x^1\sigma_x^1\sigma_x^2\sigma_y^1\sigma_y^2$$

Then, use

$(\sigma_x^i)^2 = (\sigma_y^i)^2 = (\sigma_z^i)^2 = \mathbf{1}$

repeatedly, to see that the right-hand side of the above equation equals $-\mathbf{1}$.

We now define the operators

$$C = \sigma_x^1 \sigma_y^2 \ , \quad D = \sigma_y^1 \sigma_x^2 \ , \quad E = \sigma_x^1 \sigma_x^2 \ , \quad F = \sigma_y^1 \sigma_y^2$$

$$X = CD \ , \quad Y = EF \ .$$

Using again $\sigma_\alpha^i \sigma_\beta^i = -\sigma_\beta^i \sigma_\alpha^i$,

and $\sigma_\alpha^1 \sigma_\beta^2 = \sigma_\beta^2 \sigma_\alpha^1$,

we observe:

α) $[C, D] = 0$

β) $[E, F] = 0$

γ) $[X, Y] = 0$

The identity

$$\sigma_x^1 \sigma_y^2 \sigma_y^1 \sigma_x^2 \sigma_x^1 \sigma_x^2 \sigma_y^1 \sigma_y^2 = -\sigma_y^2 \sigma_y^1 \sigma_x^2 \sigma_x^1 \sigma_x^1 \sigma_x^2 \sigma_y^1 \sigma_y^2$$

can be rewritten as

$$XY = -\mathbf{1} \ .$$

But, using $v(AB) = v(A)v(B)$ when A and B commute (assumption of the theorem) and the commutations just computed, we get:

a) $v(XY) = v(X)v(Y) = v(CD)v(EF)$

b) $v(CD) = v(C)v(D)$

c) $v(EF) = v(E)v(F)$

d) $v(C) = v(\sigma_x^1)v(\sigma_y^2)$

e) $v(D) = v(\sigma_y^1)v(\sigma_x^2)$

f) $v(E) = v(\sigma_x^1)v(\sigma_x^2)$

g) $v(F) = v(\sigma_y^1)v(\sigma_y^2)$

So,

$$v(XY) = v(X)v(Y) = v(CD)v(EF)$$

$$= v(C)v(D)v(E)v(F)$$

$$= v(\sigma_x^1)v(\sigma_y^2)v(\sigma_y^1)v(\sigma_x^2)v(\sigma_x^1)v(\sigma_x^2)v(\sigma_y^1)v(\sigma_y^2) \ ,$$

Since

$$XY = -\mathbf{1}\ .$$

and the only eigenvalue of the matrix $-\mathbf{1}$ is -1, and that, for any operator A, $v(A)$ must belong to its set of eigenvalues, we get

$$v(XY) = -1.$$

But we just saw that:

$$v(XY) =$$

$$v(\sigma_x^1)v(\sigma_y^2)v(\sigma_y^1)v(\sigma_x^2)v(\sigma_x^1)v(\sigma_x^2)v(\sigma_y^1)v(\sigma_y^2)\ ,$$

where the right-hand side equals

$$v(\sigma_x^1)^2 v(\sigma_y^2)^2 v(\sigma_y^1)^2 v(\sigma_x^2)^2,$$

since all the factors in the product appear twice. But this last expression, being the square of a real number, is positive, and so cannot equal -1.

APPENDIX 2

We will give here explicit formulas for matrices A, B, C, D used by Goldstein to prove that the different histories are inconsistent, but our presentation is due to Mermin .

Consider a basis $(|e_1\rangle, |e_2\rangle)$ of vectors for a two-dimensional spin space associated with a first particle and a basis $(|f_1\rangle, |f_2\rangle)$ associated with a second particle. Consider then the quantum state (the products between states are tensor products)

$$|\Psi\rangle = a|e_1\rangle|f_2\rangle + a|e_2\rangle|f_1\rangle - b|e_1\rangle|f_1\rangle \ , \quad (1)$$

with the normalization

$$2a^2 + b^2 = 1 \ , \quad (2)$$

where a, b will be chosen below.

Introduce the vectors

$$|g\rangle = c|e_1\rangle + d|e_2\rangle \ , \tag{3}$$

$$|h\rangle = c|f_1\rangle + d|f_2\rangle \ , \tag{4}$$

where $c^2 + d^2 = 1$ and c, d are chosen so that $\langle g f_1|\Psi\rangle = \langle e_1 h|\Psi\rangle = 0$, which means that

$$ad - bc = 0 \ . \tag{5}$$

Let P_{e_1} denote the projection operator on the vector $|e_1\rangle$, and similarly for the other vectors. We define A, B, C, D as follows:

(1) $A = P_h$,

(2) $B = P_g$,

(3) $C = P_{e_2} = \mathbf{1} - P_{e_1}$,

(4) $D = P_{f_2} = \mathbf{1} - P_{f_1}$,

where $|h\rangle = c|f_1\rangle + d|f_2\rangle$, $|g\rangle = c|e_1\rangle + d|e_2\rangle$, and in the last two identities we use the fact that $(|e_1\rangle, |e_2\rangle)$, $(|f_1\rangle, |f_2\rangle)$ are basis vectors of a two-dimensional space.

All these operators are projection operators, so their only eigenvalues are 0 and 1. We will write $A = 1$, $B = 0$ to mean that the result of the measurement of A gives 1, of B gives 0, etc. It is easy to check that all the pairs (A, C), (B, D), (C, D), and (A, B) commute, since they operate on different particles, while A does not commute with D, and B does not commute with C, since they project onto non-orthogonal vectors.

If we measure these operators when the quantum state is (**??**), we get:

(1) If $A = P_h = 1$, then since $\langle e_1 h|\Psi\rangle = 0$, we must have $P_{e_1} = 0$, and this means that $C = P_{e_2} = \mathbf{1} - P_{e_1} = 1$.

(2) If $B = P_g = 1$, then since $\langle g f_1|\Psi\rangle = 0$, we must have $P_{f_1} = 0$, which means that $D = P_{f_2} = \mathbf{1} - P_{f_1} = 1$.

(3) $CD = P_{e_2} P_{f_2} = 0$, because the state $|e_2\rangle|f_2\rangle$ is absent from the sum (**??**). Thus, if one projects $|\Psi\rangle$ onto $|f_2\rangle$, one gets $|e_1\rangle|f_2\rangle$, and P_{e_2} acting on that latter state is zero. This means that one cannot have both the results $C = P_{e_2} = 1$ and $D = P_{f_2} = 1$.

(4) $P_h = 1$ and $P_g = 1$, meaning $A = 1$, $B = 1$, has a nonzero probability of occurring. The probability is

$$|\langle gh|\Psi\rangle|^2 = |(c\langle e_1|+d\langle e_2|)(c\langle f_1|+d\langle f_2|)\Psi\rangle|^2 \tag{6}$$

$$= (2acd - bc^2)^2 = b^2c^4 \ , \tag{7}$$

where in the last equality we have used $ad = bc$ (**??**).

By $ad = bc$ and $2a^2 + b^2 = 1$ and the fact that $c^2 + d^2 = 1$, one gets $b^2 = (1 - c^2)/(1 + c^2)$, and the maximum of $(1 - c^2)c^4/(1 + c^2)$ is reached for $c^2 = (\sqrt{5} - 1)/2$ (the reciprocal of the golden mean), which yields a probability of about 9%.

Another way to state this result is that, if we want to attribute "hidden variables" taking the values 0 or 1 for each "observable" A, B, C, D and if we assume that they would be revealed by the measurement of those observables, then, if the quantum state is (**??**), we run into a contradiction. As shown by Mermin, this argument can be used to give another proof of EPR–Bell: since the two particles can be arbitrarily far from each other, if we assume no action at a distance, those hidden variables associated with the observables A, B, C, D must exist (B, C refer to the first particle and A, D to the second one) – this is the EPR part of the argument – and what we have shown here is that this assumption leads to a contradiction – the Bell part of the argument.

The decoherent histories approach would not directly claim that the four observables A, B, C, D have pre-existing values (because they do not all commute with each other), but they would attribute such values to each decoherent history corresponding to each commuting pair (A, C), (B, D), (C, D), and (A, B). But then, we get four statements that are mutually contradictory.

References:

(1) J.S. Bell: On the problem of hidden variables in quantum mechanics, *Reviews of Modern Physics* **38**, 447–452 (1966).

(2) J.S. Bell: *John Bell, Physicist*, interview by Charles Mann and Robert Crease, *Omni Magazine*, May 1988, pp. 85–92, 121

(3) J.S. Bell, *Speakable and Unspeakable in Quantum Mechanics. Collected Papers on Quantum Philosophy.*

(4) G. Birkhoff and J. von Neumann: The logic of quantum mechanics, *Annals of Mathematics* **37**, 823–843 (1936)

(5) J. Bricmont, *Making Sense of Quantum Mechanics.*

(6) R. Clifton: Complementarity between position and momentum as a consequence of Kochen–Specker arguments, *Physics Letters A* **271**, 1–7 (2000)

(7) S. Goldstein: Quantum theory without observers, *Physics Today*, Part I, March 1998, pp. 42–46; Part II, April 1998, pp. 38–42
S. Goldstein et al.: Letters and replies in *Physics Today*, February 1999, p. 11

(8) R.B. Griffiths: Consistent histories: questions and answers, available at: `quantum.phys.cmu.edu/CHS/quest.html`

(9) L. Hardy: Nonlocality for two particles without inequalities for almost all entangled states, *Physical Review Letters* **71**, 1665–1668 (1993)

(10) J. Hartle: The quantum mechanics of closed systems. *Directions in General Relativity. Proceedings of the 1993 International Symposium, Maryland: Papers in Honor of Charles Misner*, Vol. 1, B.-L. Hu, M.P. Ryan, and C.V. Vishveshwars (eds), Cambridge University Press, Cambridge, 1993, pp. 104–125

(11) D. L. Hemmick, A. M. Shakur, *Bell's theorem and quantum realism. Reassessment in light of the Schrödinger paradox*, Springer, Heidelberg, 2012

(12) S. Kochen and E. P. Specker: The problem of hidden variables in quantum mechanics, *Journal of Mathematics and Mechanics* **17**, 59–87 (1967)

(13) D. Mermin: Hidden variables and the two theorems of John Bell, *Reviews of Modern Physics* **65**, 803–815 (1993)

(14) D. Mermin: The best version of Bell's theorem, *Annals of the New York Academy of Sciences* **775**, *Fundamental Problems in Quantum Theory*, 616–623 (1995)

(15) R. Omnès: *Understanding Quantum Mechanics*, Princeton University Press, Princeton, 1999

(16) A. Peres: Two simple proofs of the Kochen–Specker theorem, *Journal of Physics A: Math. Gen.* **24**, L175–L178 (1991)

(17) W.V.O. Quine: Two dogmas of empiricism. In W.V.O. Quine: *From a Logical Point of View*, 2nd edn, revised, Harvard University Press, Cambridge Ma., 1980, first edition, 1953

(18) J.T. Schwartz: The pernicious influence of mathematics on science. In: *Logic, Methodology and Philosophy of Science*, E. Nagel, P. Suppes, A. Tarski (eds), Stanford University Press, Stanford, 1962

(19) Tumulka, R. Lecture Notes on Foundations of Quantum Mechanics https://www.math.uni-tuebingen.de/de/forschung/maphy/personen/roderichtumulka/lehre-ws2017-18/quantmech/dateien/script2017.pdf

(20) J. von Neumann, *Mathematical Foundations of Quantum Mechanics*, Princeton University Press, Princeton, 1955.