Quantum Zeno's Paradox

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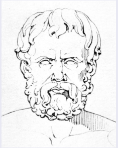
Ancient Zeno's paradoxes



Zeno of Elea (c. 490–c. 430 BCE) claimed that motion and time cannot exist because they are inherently paradoxical notions. He formulated various paradoxes such as this one:

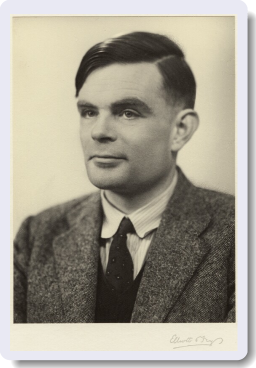
Paradox of Achilles and the tortoise

Achilles, a character of ancient Greek poet Homer (c. 1,000 BCE) known for being a fast runner, competes at a race against a tortoise, an animal known for being slow. The tortoise gets a head start of 10 meters.
Claim: Achilles will never pass the tortoise!
Argument: Suppose Achilles is 10 times faster. When he has finished the 10 m, the tortoise will have completed 1 m. When Achilles is done with that meter, the tortoise has added 10 cm, and so on—the tortoise is always ahead!

Mathematician Harro Heuser commented in his 1980 math textbook: "Basically, Zeno could not believe that an infinite series could converge." (Achilles passes at $10 + \sum_{n=0}^{\infty} 10^{-n} = \frac{100}{9}$ meters.)

Quantum Zeno's paradox

- The name was coined by Misra and Sudarshan [J.Math.Phys. 1977], who proved that the "quantum Zeno effect" follows rigorously from quantum mechanics under weak and general assumptions.
- The paradox was discovered by computer scientist Alan Turing (1912–1954) in 1954.
- It arose from trying to compute the probability distribution of the time at which a detector clicks.
- So let's talk about that.



Born's rule on spacelike and timelike surfaces

Born's rule on a horizontal surface in space-time

If we place ideal detectors along $\{t = \text{const.}\}$, then the place of detection has probability distribution $\rho(\boldsymbol{x})\, d^3\boldsymbol{x} = |\psi_t(\boldsymbol{x})|^2\, d^3\boldsymbol{x}$

Born's rule on a spacelike surface Σ in relativistic space-time

If we place detectors along Σ, then the place of detecting a Dirac particle has probability distribution

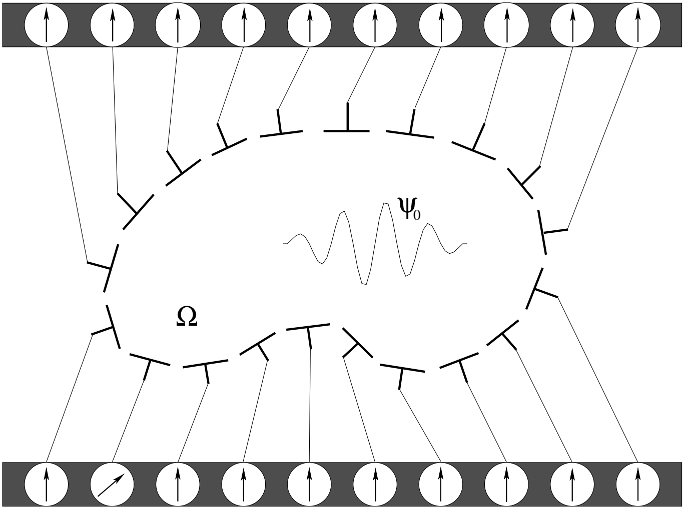
$\rho(x)\, V(d^3x) = \rho(x)\, \sqrt{\det {}^{(3)}g}\, d^3x = \overline{\psi}(x)\, u_\mu(x)\, \gamma^\mu\, \psi(x)\, V(d^3x)$

with $u(x)$ = future unit normal on Σ at x. [Lienert, Tumulka arXiv:1706.07074]

Question

Is there a Born rule also for a timelike surface? In the relativistic or non-relativistic case? A simple formula for the distribution of the time and place of arrival at waiting detectors?

Problem of detection time and place



$T \in [0, \infty), \boldsymbol{X} \in \partial\Omega, Z = (T, \boldsymbol{X})$

Picture: redrawn after Detlef Dürr

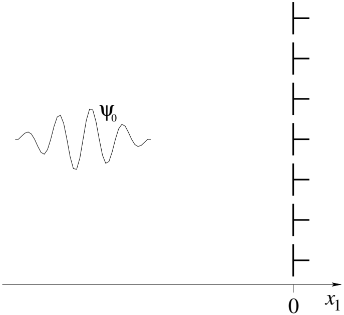
Problem of detection time and place

- $\Omega \subset \mathbb{R}^3$, $\psi_0 \in L^2(\Omega, \mathbb{C})$, detecting surface $\partial\Omega$
- $Z = (T, \boldsymbol{X})$, or $Z = \infty$ if no detector ever clicks
- Problem: Compute the distribution of Z from ψ_0.
- This is different from computing, in Bohmian mechanics, the time and place at which the particle first exits Ω in the absence of detectors. (Thus, "time of arrival" is an ambiguous name, "time of detection" is better.)
- Hurdle: QM does not provide a self-adjoint time operator.

Historical proposals for the detection time distribution

- **Pauli 1958:** There is no time operator $\hat{T}$ because it would have to be canonically conjugate to the Hamiltonian, $[H, \hat{T}] = i\hbar$, which doesn't exist.
- **Aharonov and Bohm 1961:** Compute the classical time of arrival $T_{\mathrm{cl}}(\boldsymbol{q}, \boldsymbol{p})$ at $\partial\Omega$ (e.g., $T_{\mathrm{cl}} = -mq_1/p_1$),
  then "quantize" the formula (e.g., $\hat{T} = -m\hat{p}_1^{-1/2}\hat{q}_1\hat{p}_1^{-1/2}$).
  Problem: has nothing to do with how detectors work.
- **Allcock 1969:** complex potentials
  Good but provides only "soft" detectors. (See later.)
- **Kijowski 1974:** Assume there is a self-adjoint arrival-time observable $\hat{T}$, make axioms for its properties (e.g., symmetries), then find such operators.
- **Leavens 1996:** Distribution of the time at which the Bohmian trajectory in the absence of detectors would arrive on $\partial\Omega$.
  Problem: does not take the presence of the detectors into account.

Turing's approach to detection time



- Say, $\Omega = \{x_1 \leq 0\}$ and $\partial\Omega =$ plane $\{x_1 = 0\}$.
- Make an instantaneous quantum measurement of the event $x_1 > 0$ (the projection operator $1_{x_1>0}$) at regular time intervals $\tau > 0$.
- Consider the limit $\tau \to 0$.
- Result: In the limit, the probability of ever finding $x_1 > 0$ becomes 0.

- This is called the quantum Zeno effect.
- It seems to make any concept of ideal detector impossible.
- Misra and Sudarshan: "A watched pot never boils."

Derivation of the effect in a simple case

- In a 2d Hilbert space $\mathbb{C}^2$, let $\psi_0 = (1,0)$ evolve with Hamiltonian $H = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, interrupted by a quantum measurement of $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ at times $n\tau$ for all $n \in \mathbb{N}$.
- For any fixed $T > 0$, the probability that any of the $\approx T/\tau$ measurements in the time interval $[0, T]$ yields the result -1 tends to 0 as $\tau \to 0$.

Proof: Unitary evolution yields $\psi(\tau) = (\cos\tau, -i\sin\tau)$, then measure σ_3.
$\mathbb{P}(\text{“no detection”}) = \mathbb{P}(\text{eigenvalue} + 1) = \cos^2\tau$,
then collapses back to $(1,0)$.
Prob of no detection in m consecutive trials is $p = (\cos\tau)^{2m}$. Let $\tau \to 0$, $m \to \infty$ so that $m\tau \to T$. Since $1 - \tau^2/2 \le \cos\tau \le 1$,

$$1 \ge p \ge \left(1 - \frac{\tau^2}{2}\right)^{2m} = 1 - 2m\frac{\tau^2}{2} + \binom{2m}{2}\frac{\tau^4}{2^2} + \ldots \longrightarrow 1\,.$$

$\square$

“A watched pot never boils.”

Soft detectors

- A soft detector is one which takes a while to detect a particle in the detector volume.
- A hard detector is one which registers the particle immediately.
- A soft detector can be modeled by an imaginary potential:
- E.g., $\Omega = \{x_1 \leq 0\}$ and $\partial\Omega =$ plane $\{x_1 = 0\}$.
- Schrödinger equation with complex potential

$$V(\boldsymbol{x}) = \begin{cases} -iv & \text{if } x_1 > 0 \\ 0 & \text{if } x_1 \leq 0\,, \end{cases}$$

where $v > 0$ is a constant.

- Leads to continuity eq

$$\frac{\partial|\psi(\boldsymbol{x})|^2}{\partial t} = -\nabla \cdot \boldsymbol{j} - \frac{2v}{\hbar} 1_{x_1>0}$$

- This means that in the right half space the particle has rate $2v/\hbar$ of being absorbed (loss of $\|\psi\|^2$). Non-unitary. The Bohmian particle disappears (gets detected and absorbed) at a random time T. $\mathrm{Prob}(\boldsymbol{X} \in d^3\boldsymbol{x} | T) = |\psi_T(\boldsymbol{x})|^2 d^3\boldsymbol{x}$.

Allcock's paradox [Ann.Phys. 1969]

- Average lifetime in the detector volume $= \hbar/2v$.
- Allcock's approach to the ideal hard detector:
- Consider a soft detector, take the hard limit $v \to \infty$.
- Difficulty: In the limit, $\psi_t(\boldsymbol{x}) = 0$ for $x_1 > 0$ and all $t > 0$, so the particle never gets detected.
- Time evolution equivalent to Dirichlet boundary condition $\psi(0, x_2, x_3) = 0$. Waves arriving from the left get completely reflected to the left.
- Again, a hard ideal detector seems impossible, in line with the quantum Zeno effect.

Derivation of Allcock's paradox

- In a 2d Hilbert space $\mathbb{C}^2$, let $\psi_0 = (1,0)$ evolve with the (non-self-adjoint) Hamiltonian

$$H_v = \begin{pmatrix} 0 & 1 \\ 1 & -iv \end{pmatrix} .$$

- Then for every $t > 0$, $\psi_t = e^{-iH_v t/\hbar}\psi_0 \to \psi_0$ as $v \to \infty$.

Proof: Calculation.

Does QM make a prediction?

- Although QM does not provide a self-adjoint time operator, it makes an unambiguous prediction for the distribution of Z (though in an un-orthodox way): Solve the Schrödinger equation of the big system formed by "the" particle, all detectors, a clock, and a recording device, constructed so as to keep a record of which detector clicked when. At a late time t, make a quantum measurement of the record.
- It follows that the distribution of Z is given by a continuous POVM,

$$\mathrm{Prob}_{\psi_0}(Z \in \Delta) = \langle \psi_0 | E(\Delta) | \psi_0 \rangle \,.$$

POVM (positive-operator-valued measure) on $\mathscr{Z}$

Def: For every (measurable) set $\Delta \subseteq \mathscr{Z}$, $E(\Delta)$ is a positive operator. $E(\mathscr{Z}) = I$, and $E(\Delta_1 \cup \Delta_2 \cup \ldots) = E(\Delta_1) + E(\Delta_2) + \ldots$ if $\Delta_1, \Delta_2, \ldots$ are mutually disjoint.

- Is there a practical way of computing $E(\cdot)$, at least approximately? Without solving a Schrödinger equation for $> 10^{23}$ particles?

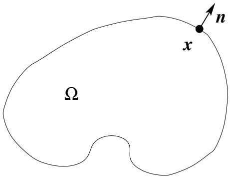
- While the correct POVM $E(\cdot)$ will depend on all details of the detectors, including their quantum states at time 0, the expectation is that there is a particular POVM E_0 (or maybe E_κ depending on one or few parameters κ) in the cloud of E's that is a good approximation and can be expressed by some simple rule. E_0 represents an ideal detector.
- I will describe a POVM E_κ for that.
- That is, I will show you how an ideal hard detector is possible.
- So a watched pot does boil, after all!
- Although the quantum Zeno effect is for real.

Proposed solution: The "absorbing boundary rule"

- Solve the 1-particle Schrödinger equation $i\hbar\dfrac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi$ with "absorbing boundary condition" (ABC)

$$\boldsymbol{n}(\boldsymbol{x})\cdot\nabla\psi(\boldsymbol{x}) = i\kappa\,\psi(\boldsymbol{x})$$

at every $\boldsymbol{x}\in\partial\Omega$, where $\boldsymbol{n}(\boldsymbol{x}) =$ outward unit normal vector to $\partial\Omega$ at $\boldsymbol{x}$, and $\kappa > 0$ a constant.



- ABC implies that the probability current $\boldsymbol{j}^\psi = \frac{\hbar}{m}\mathrm{Im}[\psi^*\nabla\psi]$ points outward at $\partial\Omega$:

$$\boldsymbol{n}\cdot\boldsymbol{j} = \frac{\hbar}{m}\mathrm{Im}[\psi^*\boldsymbol{n}\cdot\nabla\psi] = \frac{\hbar}{m}\mathrm{Im}[\psi^* i\kappa\psi] = \frac{\hbar}{m}\kappa|\psi|^2 \geq 0\,.$$

- $\mathrm{Prob}_{\psi_0}\Big(T\in dt, \boldsymbol{X}\in d^2\boldsymbol{x}\Big) = \boldsymbol{n}(\boldsymbol{x})\cdot\boldsymbol{j}^{\psi_t}(\boldsymbol{x})\,dt\,d^2\boldsymbol{x}$ assuming $\|\psi_0\| = 1$.
- If the experiments get interrupted at time t before detection, the collapsed wave function is $\psi_t/\|\psi_t\|$.

Properties

- $\|\psi_t\|^2 = \mathrm{Prob}_{\psi_0}(T > t)$ "survival probability," decreasing in t
- The time evolution of ψ is not unitary (Hamiltonian not self-adjoint) due to loss at $\partial\Omega$, but well defined by the Hille-Yosida theorem: $\psi_t = W_t \psi_0$ with $W_t = e^{-iHt/\hbar}$ a contraction semigroup ($t \geq 0$) on $L^2(\Omega)$.
- skew-adjoint part(H) is a negative operator, i.e., $\mathrm{Im}\langle\psi|H\psi\rangle \leq 0$.
- H is not necessarily diagonalizable; if it is, then spectrum $\subseteq \{x + iy \in \mathbb{C} : y \leq 0\}$ = lower half plane.
- In Bohmian mechanics, the particle with $|\psi_0|^2$-distributed initial condition $\boldsymbol{X}(0)$ moves according to the equation of motion

$$\frac{d\boldsymbol{X}}{dt} = \frac{\boldsymbol{j}^{\psi_t}(\boldsymbol{X}(t))}{|\psi_t(\boldsymbol{X}(t))|^2}$$

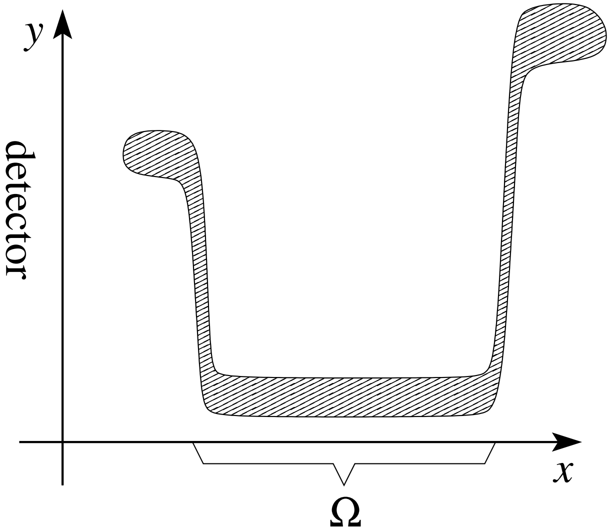
  until it hits $\partial\Omega$ at time T and place $\boldsymbol{X} = \boldsymbol{X}(T)$, and gets absorbed. $\mathrm{Prob}_{\psi_0}\big(\boldsymbol{X}(t) \in d^3\boldsymbol{x}\big) = |\psi_t(\boldsymbol{x})|^2 \, d^3\boldsymbol{x}$.
- energy-time uncertainty relation $\Delta E \, \Delta T \geq \hbar/2$ with E referring to $-\frac{\hbar^2}{2m}\nabla^2$ on $L^2(\mathbb{R}^3)$

POVM

- $E_\kappa\big(dt \times d^2\boldsymbol{x}\big) = \frac{\hbar\kappa}{m}\, W_t^\dagger\, |\boldsymbol{x}\rangle\langle\boldsymbol{x}|\, W_t\, dt\, d^2\boldsymbol{x}$,
  $E_\kappa(T=\infty) = \lim_{t\to\infty} W_t^\dagger W_t$
- on $\mathscr{Z} = [0,\infty) \times \partial\Omega \cup \{\infty\}$, acting on $L^2(\Omega)$
- not PVM $\Rightarrow$ no "eigenstates of detection time"

Why to expect an absorbing boundary

configuration space:



Literature

- The ABC was considered by Werner in 1987 [J. Math. Phys.], indeed for detection time distribution ("any contraction semigroup determines a natural arrival time observable").
- Afterward [1988], Werner studied less compelling approaches to the detection time distribution.
- The ABC received almost no attention. In an 86-pages review paper [Muga and Leavens, Phys. Rep. 2000], the ABC was mentioned in passing but not even written down.
- The ABC was mentioned by Fevens and Jiang in 1999 [SIAM J. Sci. Comput.] for numerical simulation of the Schrödinger eq. on $\mathbb{R}$ with finitely many lattice points, but dropped in favor of a higher-order BC that absorbs more of the wave.
- Recent investigations: Tumulka [arXiv:1601.03715, 1601.03871]

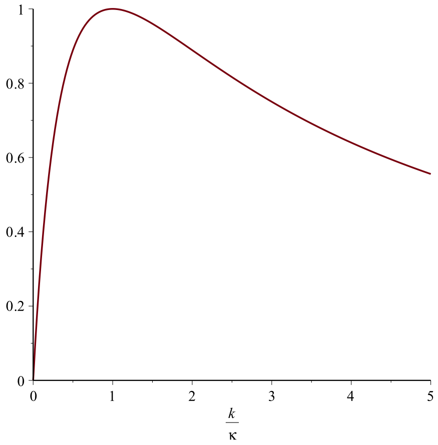
Reflection from $\partial\Omega$

While the Bohmian particle always gets absorbed when it hits $\partial\Omega$, the wave function gets partly reflected and partly absorbed.

Absorption coeff. $A_k = 1 - R_k$, reflection coefficient $R_k = |c_k|^2$ for eigenfct $\psi(x) = e^{ikx} + c_k e^{-ikx}$ satisfying ABC $\psi'(0) = i\kappa\psi(0)$ in 1D.

κ = wave number of maximal absorption

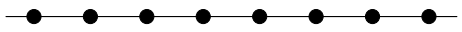
Corollary: The presence of the detectors changes the Bohmian trajectories even before they reach $\partial\Omega$.



Plot of A_k

Lattice version (e.g., in 1D)

- lattice of mesh width $\varepsilon > 0$: $\Lambda = \{\varepsilon, 2\varepsilon, 3\varepsilon, \ldots, N\varepsilon\}$
- $\mathscr{H} = L^2(\Lambda) = \mathbb{C}^N$



- Hamiltonian = discrete Laplacian, $H = -(\hbar^2/2m\varepsilon^2)\times$

$$\begin{bmatrix} -1 & 1 & & & & \\ 1 & -2 & 1 & & & \\ & 1 & -2 & 1 & & \\ & & 1 & -2 & 1 & \\ & & & 1 & -2 & 1 \\ & & & & 1 & -1 \end{bmatrix} \text{ or } \begin{bmatrix} i\kappa\varepsilon - 1 & 1 & & & & \\ 1 & -2 & 1 & & & \\ & 1 & -2 & 1 & & \\ & & 1 & -2 & 1 & \\ & & & 1 & -2 & 1 \\ & & & & 1 & i\kappa\varepsilon - 1 \end{bmatrix}$$

Neumann b.c. absorbing b.c.

- H not self-adjoint, $W_t = e^{-iHt/\hbar}$ contraction semigroup $(t \geq 0)$
- $\mathrm{Prob}_{\psi_0}(T \in dt, \boldsymbol{X} = N\varepsilon) = \frac{\hbar\kappa}{m\varepsilon}|\psi_t(N\varepsilon)|^2$

Avoiding the quantum Zeno effect

- Again, lattice of mesh width ε, $\Lambda = \{\varepsilon, 2\varepsilon, 3\varepsilon, \dots, N\varepsilon\}$
- Neumann b.c.
- quantum measurement of $P = |N\varepsilon\rangle\langle N\varepsilon|$ at times $\tau, 2\tau, 3\tau, \dots$
- quantum Zeno effect occurs in the limit $\tau \to 0$, $\varepsilon =$ const., $N =$ const.
- The limit $\tau \to 0$, $\varepsilon \to 0$, $N \to \infty$, $N\varepsilon \to L$, $\tau/\varepsilon^3 \to 4m\kappa/\hbar$ leads to the absorbing boundary rule (no quantum Zeno effect!).
- Thus, a non-trivial limit is possible.

Avoiding Allcock's difficulty

- Again, $\Omega = \{x_1 \leq 0\}$ and $\partial\Omega =$ plane $\{x_1 = 0\}$.
- Different model of soft detector:
- Consider Schrödinger equation in $\{x_1 \leq L\}$ with complex potential

$$V(\boldsymbol{x}) = \begin{cases} -iv & \text{if } x_1 > 0 \\ 0 & \text{if } x_1 \leq 0 \end{cases}$$

  and Neumann boundary condition

$$\frac{\partial\psi}{\partial x_1}(L, x_2, x_3) = 0\,.$$

- The hard limit $v \to \infty, \quad L \to 0, \quad vL \to \frac{\hbar^2\kappa}{2m} > 0$ leads to the absorbing boundary rule.
- Thus, a non-trivial hard limit is possible.

Further developments

- continuum limit of the lattice version reproduces the continuum version
- rigorous existence of ψ_t, Z, E_κ
- version of the rule for moving detectors
- version of the rule for several particles, in particular how to collapse ψ after the first detection
- version of the rule for particles with spin
- one may measure a spin component simultaneously with the detection
- version of the rule for the Dirac equation
- non-relativistic limit of the Dirac equation with ABC
- version of the rule in curved space-time
- boundary may be partly spacelike and partly timelike
- formulation in terms of multi-time wave functions for n particles
- ...so the absorbing boundary rule is very robust!

Thank you for your attention