THE DE BROGLIE-BOHM THEORY AS A RATIONAL COMPLETION OF QUANTUM MECHANICS [1]

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Abstract

We try to give a physical meaning to the wave function or quantum state of a system, apart from being a very efficient tool for predicting results of measurements on that system. In other words, we ask: what does it mean for a system outside the laboratories to have a wave function?

We first explain why two possible, and probably common, answers to that question do not really answer it.

Then, we explain how the de Broglie-Bohm theory does give a satisfactory meaning to the quantum state outside the laboratories, while avoiding the problems faced by the other answers.

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1 Introduction

Quantum theory is probably the most spectacularly successful scientific theory that exists. It is at the basis of many technologies and its predictions have never been invalidated by experiments.

Yet, what this theory means has been debated since its origins and still is. The simplest way to raise the basic question is: what does it mean for a physical system to have or to be represented by a quantum state?

Namely by a vector Ψ of unit norm $\|\Psi\| = 1$ in a Hilbert space!

The orthodox answer is that, on the one hand, this state evolves according to Schrödinger's equation (see (18) below) or a similar equation, as long as one does not perform a measurement on the system. This evolution is *continuous in time, deterministic and linear.*

But things change radically when one performs measurements. Consider a system, whose quantum state is Ψ. Let us measure on that system a quantity represented by a self-adjoint operator A, having a basis of eigenvectors[2] Ψ_n, with eigenvalues λ_n. Then, one obtains the result λ_k with probability $|c_k|^2$, where:

$$\Psi = \sum_n c_n \Psi_n \tag{1}$$

is the expansion of Ψ in the basis (Ψ_n). Since (1) is an expansion in an orthonormal basis, and $\|\Psi\| = 1$, one has $\sum_n |c_n|^2 = 1$, which means that one can interpret each $|c_k|^2$ as a probability.

Moreover, after the measurement of the observable represented by the operator A, the state "jumps" or "is reduced" to Ψ_k.

This latter operation is *discontinuous in time* (the state "jumps"), *random* (one only gives probabilities $|c_k|^2$ for those jumps), and *non linear* (the result Ψ_k is independent of the coefficients c_n in (1)).

[2] We put aside here the issue of continuous spectra, that can be treated similarly from a conceptual viewpoint.

The problem is that the only meaning given to the quantum state depends entirely on measurements; if one asks what a state attributed to a system means outside the laboratories, then no clear answer is given in standard quantum mechanical textbooks.

But what happened during evolution? Biological processes rely in principle on chemistry that itself is ultimately based on quantum mechanics. But how does one describe quantum processes before the appearance of human observers?

This problem was expressed ironically by Bell:

> It would seem that the theory is exclusively concerned about "results of measurement", and has nothing to say about anything else. What exactly qualifies some physical systems to play the role of "measurer"? Was the wavefunction of the world waiting to jump for thousands of millions of years until a single-celled living creature appeared? Or did it have to wait a little longer, for some better qualified system... with a Ph D?
>
> John S. Bell [6, p. 34]

Does physics really have nothing to say about the world outside the laboratories? And if that was the case, why build laboratories in the first place? Experiments are made to test our theories about the world, but those theories are not about experiments alone. Again, Bells says it clearly:

> But experiment is a tool. The aim remains: to understand the world. To restrict quantum mechanics to be exclusively about piddling laboratory operations is to betray the great enterprise. A serious formulation [of quantum mechanics] will not exclude the big world outside the laboratory.
>
> John Bell [6, p. 34]

Moreover, how is one supposed to understand this duality of rules in the evolution of a quantum state: a continuous in time, deterministic and linear evolution outside of measurements and a discontinuous in time, random and nonlinear evolution during measurements?

We will first discuss two answers to those questions, one of which is probably in the back of the mind of the physicists who think that quantum mechanics does not pose any conceptual problem.

After having explained why none of these answers is satisfying, we will explain that the real problem of quantum mechanics is even deeper than the fact that those answers do not work.

Finally, we will introduce a more detailed description of physical systems than the one given by the quantum state and that will solve the conceptual problems of quantum mechanics: the de Broglie-Bohm theory.

2 Two “solutions” that do not solve anything

The first “solution” is to consider the measuring process and treat the measuring device in a quantum mechanical way, hoping that this would eliminate the duality of rules and in particular make the reduction appear as a consequence of the temporal evolution of the state of the system composed of the system being measured and the measuring device. After all, the latter is necessarily macroscopic and the measured system microscopic. Could not the huge difference of scale between the two allow us to solve the problem? Isn’t this analogous to the fact that microscopic laws are time reversible while the macroscopic laws are not and, yet, statistical mechanics explains how the latter emerges on the basis of the former[3]?

Could the reduction of the quantum state be a similar phenomenon?

Unfortunately, that is not the case[4].

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[3] See for example [25].

[4] We will see in section 3.2 that one can make an analogy between the reduction of the quantum state and macroscopic irreversibility, but only within a more complete theory than ordinary quantum mechanics.

2.1 A Quantum Analysis of the Measuring Device

Let us consider a very simple quantum system, composed of the spin of a single particle (and we will neglect here the wave function, that is the spatial part of the quantum state), whose initial state is:

$$c_1|1\uparrow\rangle + c_2|1\downarrow\rangle, \tag{2}$$

where $|1\uparrow\rangle$ denotes the "spin up" state in a given direction labelled 1 (and $|1\downarrow\rangle$ denotes the "spin down" state in that direction); c_1, c_2 are complex numbers. Formula (2) is a special case of (1), and one has therefore $|c_1|^2 + |c_2|^2 = 1$.

The measuring device is represented in figures 1–3, and is reduced to a pointer, which is initially in a horizontal position and, at the end of the measurement, is either pointing up or down, depending on whether the spin after it has been measured is "up" or "down".

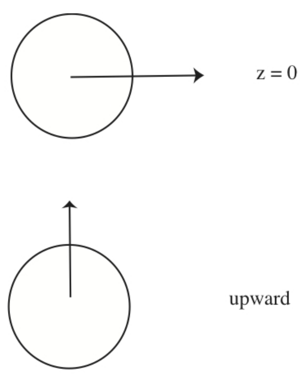

Fig. 1: Evolution of the pointer during the measurement when the initial state is given by (4)

In order to describe the measuring device quantum mechanically, one must associate to it a quantum state. Let $\varphi_0(z)$ be the initial wave function associated to the pointer, with $\varphi_0(z)$ centered at $z = 0$, which means that the pointer is as in the first picture of figure 1.

Let:

$$\Psi_0 = \varphi_0(z)\,[c_1|1\uparrow\rangle + c_2|1\downarrow\rangle]\,, \tag{3}$$

be the quantum state that describes the initial state of the system composed of the spin of the particle and of the pointer.

Consider first another initial state, corresponding to the particle spin being “up”:

$$\Psi_0^{\uparrow} = \varphi_0(z)|1\uparrow\rangle. \tag{4}$$

Then, since the pointer will go up if the spin is up, the final state will be:

$$\varphi^{\uparrow}(z)|1\uparrow\rangle,$$

where $\varphi^{\uparrow}(z)$ corresponds to the pointer being as in the second image of figure 1.

Similarly, if one starts from the initial state:

$$\Psi_0^{\downarrow} = \varphi_0(z)|1\downarrow\rangle, \tag{5}$$

the final state will be:

$$\varphi^{\downarrow}(z)|1\downarrow\rangle,$$

where $\varphi^{\downarrow}(z)$ corresponds to the pointer being as in the second image of figure 2.

Let us see what happens if the initial state is Ψ_0 (see (3)). Since this state is a linear combination of two other states, $\Psi_0^{\uparrow}$ (see (4)) and $\Psi_0^{\downarrow}$ (see (5)), and, since the evolution is linear, the final result will necessarily be:

$$c_1\varphi^{\uparrow}(z)|1\uparrow\rangle + c_2\varphi^{\downarrow}(z)|1\downarrow\rangle. \tag{6}$$

Concerning the pointer, one can only interpret this state as being a “superposition” of two different macroscopic states: one where the pointer points upwards $\varphi^{\uparrow}(z)$ and another where it point downwards $\varphi^{\downarrow}(z)$. This is illustrated symbolically in figure 3.

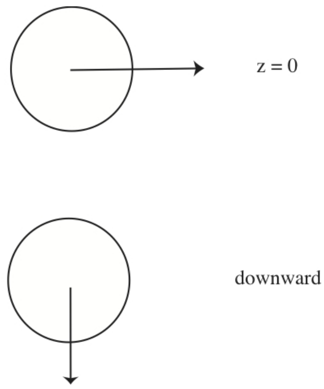

Fig. 2: Evolution of the pointer during the measurement when the initial state is given by (5)

The problem is that (6) does not at all represent the state of the measuring device as we know it. The pointer points either upwards or downwards, but is not a superposition of both! Or, if one prefers, the *complete* description of the measuring device after the measurement is certainly not a superposition, since a more complete description (either upwards or downwards) can be obtained simply by looking at the result. The problem is that measurements have well defined results and the quantum formalism does not at all account for this fact.

It is sometimes suggested that the problem mentioned here is solved by the fact that it is impossible to produce interference effects between a state of the pointer oriented upwards and the one oriented downwards comparable to the effects that one can produce with systems composed of a few particles. This is true (and is called decoherence) but it does not solve the problem, namely that quantum mechanics does not account for the uniqueness of outcomes (the pointer being up or down, not both). What decoherence implies is that when one observes the pointer being up or down, one can use that state, and forget about the other one, in order to predict the future behavior of the system, without risk of making mistakes. But the conceptual problem, namely the central role of

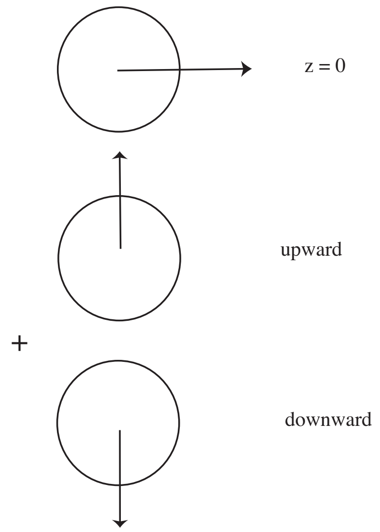

Fig. 3: Evolution of the pointer during the measurement when the initial state is given by (3)

the observer, subsists[5].

To make the situation look more dramatical, one can couple the apparatus to a cat, as suggested by Schrödinger [30]: one imagines a cat in a sealed box with a mechanism that links the position of the pointer to a hammer that breaks a vial of poison if the pointer is oriented upwards and does not break it if the pointer is oriented downwards. If the vial is broken, the poison kills the cat (see figure 4). Reasoning as above, the complete state of the system, including the cat becomes:

$$c_1\varphi^{\uparrow}(z)|1\uparrow\rangle|\text{dead cat}\rangle + c_2\varphi^{\downarrow}(z)|1\downarrow\rangle|\text{live cat}\rangle. \tag{7}$$

One obtains again a macroscopic superposition given by a "sum" of a live cat and a dead cat, expression that certainly does not correspond to reality and to which it is difficult to even ascribe a meaning. To say things differently, quantum mechanics is obviously not complete since, after the measurement, one can describe physical systems in a more complete way that what quantum mechanics does.

Giving a more complete description than the quantum mechanical is usually called "introducing hidden variables" (the latter, by definition, refer to anything in the descrip-

[5] We will see in section 3.2 that decoherence is important in order to understand the de Broglie-Bohm theory.

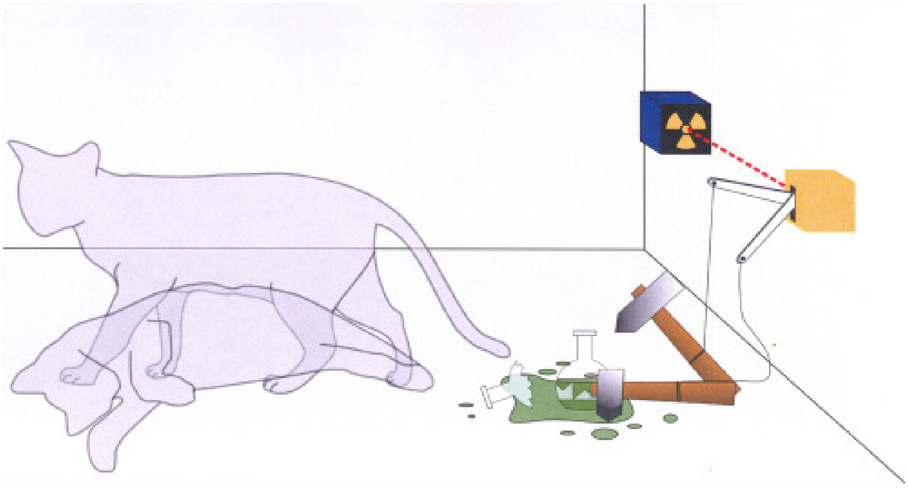

Fig. 4: The cat that is both dead and alive. By Dhatfield (own work) [CC BY-SA 3.0 (`http://creativecommons.org/licenses/by-sa/3.0`)], via Wikimedia Commons

tion of a physical system that is not included in the quantum state). Of course, in the case of the cat, the fact that she is alive or dead in not at all hidden.

But one may suggest that "hidden variables" exist at all levels: not only for the dead or live cat, for the pointer going up or down, but also for the particle whose spin is measured. One could think that a state such as (2) means that the spin has a probability $|c_1|^2$ to be "up" and a probability $|c_2|^2$ to be "down" and that the measurement simply reveals that value of the spin. More precisely, one could think that, if one prepares and large number of physical systems characterized by the state (2), then a fraction of them equal to $|c_1|^2$ will have its spin "up" and a fraction equal to $|c_2|^2$ will have its spin "down".

That is what one sometimes calls the statistical interpretation of the quantum state. If one adopts that view, the reduction of the state is not so mysterious, since a measurement would simply tell us a property of the system and, as in classical probability, learning something about a system makes us change the probabilities assigned to that system: for example, if a coin falls on one of its faces, but we do not know which one, we assign a probability $(1/2, 1/2)$ to each face; but, if we learn on which face it fell, then we change our probabilities and give a probability equal to one for the face on which it fell and zero for the other one.

This way of thinking is extremely natural: if one says that a measurement measures something, then one thinks that what is measured pre-exist to its measurement: if I measure the length of a table, I assume that the table *has* a certain length before I measure it; otherwise, what could the word “measure” possibly mean?

This way of thinking was probably what Einstein had in mind when he summarized his position about quantum mechanics in 1949:

> I am, in fact, firmly convinced that the essentially statistical character of contemporary quantum theory is solely to be ascribed to the fact that this (theory) operates with an incomplete description of physical systems [...].
>
> Albert Einstein [16, p. 666]

If the statistical interpretation of the quantum state could work, all our problems would be solved. Unfortunately, there exist relatively unknown mathematical theorems that show that it cannot work.

2.2 The Impossibility of a Statistical Interpretation of the Quantum State

Let us define the idea of “hidden variables” more precisely. There are many physical quantities besides spin that can in principle be measured: for example, the angular momentum, the energy, or the momentum. The statistical interpretation means that, in each individual system, each of these quantities will have a well defined value, which may be unknown or even unknowable, and uncontrollable, but which nevertheless *exists*.

Let us denote by X a physical quantity (identified with the self-adjoint operator to which it is associated) and by $v(X)$ the value that this quantity has for a particular system, which of course varies from system to system, but in such a way that the quantum state gives the statistical distribution of those values. By definition, such values $v(X)$ are called “hidden variables” (although we will see below that, in the de Broglie-Bohm theory, one introduces hidden variables that are not at all hidden).

To make the statistical interpretation interesting, we have to assume that $v(X)$ exists for more than one X. For example, it would be quite arbitrary to assume that the spin values exist, but only in one direction, since our definition of directions is completely conventional. Now, if we assume that $v(X)$ exists for a reasonable class of quantities X, and that those values agree with quantum mechanical predictions, we can derive a contradiction.

The function $v(X)$ must be such that it coincides with one of the possible results obtained when measuring the quantity X, which implies that the function $v(X)$ must satisfy the following properties:

$$v(X) \in \{\text{spectrum of the operator X}\} \tag{8}$$

and, $\forall X, Y$, if $[X, Y] = XY - YX = 0$, then:

$$v(XY) = v(X)v(Y). \tag{9}$$

Indeed (8) simply says that if a measurement reveals a pre-existing value of an operator[6] X (which, we will assume, has a discrete spectrum), $v(X)$ must be one of the eigenvalues of that operator (for example, if X is the spin discussed here, the value taken by $v(X)$ will be "up" or "down", or $+1$ ou -1, if one wants to associate numbers to "up" and "down").

Actually, the proof of the theorem below will only use a very special case of (8):

$$v(-\mathbf{1}) = -1 \tag{10}$$

where $\mathbf{1}$ is the unit operator.

As for (9), it expresses also a quantum mechanical constraint if measurements are to reveal pre-existing properties. Indeed, if operators X and Y commute, then one can measure both quantities without perturbing the obtained values; one can also measure the quantity XY and the corresponding values must satisfy (9).

[6] Although we speak of operators here, we will only use finite dimensional ones in the theorem below.

So, the constraints (8) and (9) may be considered as purely empirical. They do not depend on the fact that quantum mechanics is a correct theory, but only on certain of its predictions that have been very well verified experimentally.

However, and here is the problem, one can easily prove the:

Theorem on the inexistence of "hidden variables"[7]

There does not exist a function $v : \mathcal{A} \to \mathbb{R}$

where $\mathcal{A}$ = set of self-adjoint matrices on a vector space of dimension at least equal to 3, such that $\forall\, X, Y \in \mathcal{A}$, the constraints (10) and (9) are satisfied.

Proof[8]

We use the standard self-adjoint Pauli matrices σ_x and σ_y:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

We consider a couple of each of those matrices, σ_x^i, σ_y^i, $i = 1, 2$, where tensor products are implicit: $\sigma_x^1 \equiv \sigma_x^1 \otimes \mathbf{1}$, $\sigma_x^2 \equiv \mathbf{1} \otimes \sigma_x^2$, etc., with $\mathbf{1}$ the unit matrix. These operators act on $\mathbf{C}^4$. The following identities are well known and easy to check:

i)

$$(\sigma_x^i)^2 = (\sigma_y^i)^2 = \mathbf{1}, \tag{11}$$

for $i = 1, 2$.

ii) Different Pauli matrices anticommute:

$$\sigma_x^i \sigma_y^i = -\sigma_y^i \sigma_x^i, \tag{12}$$

for $i = 1, 2$.

[7] The original version of this theorem is due to Bell [5] and to Kochen and Specker [21] (the proof of Bell was based on a theorem of Gleason [19]). The version given here is simpler than the original ones and is due to Mermin (see [26] and reference therein) and Perez [27, 28]. See [10], section 2.5 for more details.

[8] The proof given here is only valid in dimension 4 and can be easily extended to spaces whose dimension is a multiple of 4. See [5, 21, 26] and [10], Appendix 2.F for more details on the general case.

iii) Finally,

$$[\sigma_\alpha^1, \sigma_\beta^2] = \sigma_\alpha^1 \sigma_\beta^2 - \sigma_\beta^2 \sigma_\alpha^1 = \mathbf{0}, \tag{13}$$

where $\alpha, \beta = x, y$ and $\mathbf{0}$ is the matrix with all entries equal to zero.

Consider now the identity

$$\sigma_x^1 \sigma_y^2 \sigma_y^1 \sigma_x^2 \sigma_x^1 \sigma_x^2 \sigma_y^1 \sigma_y^2 = -\mathbf{1}, \tag{14}$$

which follows, using first ii) and iii) above to move σ_x^1 in the product from the first place (starting from the left) to the fourth place, a move that involves one anticommutation (12) and two commutations (13), viz.,

$$\sigma_x^1 \sigma_y^2 \sigma_y^1 \sigma_x^2 \sigma_x^1 \sigma_x^2 \sigma_y^1 \sigma_y^2 = -\sigma_y^2 \sigma_y^1 \sigma_x^2 \sigma_x^1 \sigma_x^1 \sigma_x^2 \sigma_y^1 \sigma_y^2, \tag{15}$$

and then using i) repeatedly, to see that the right-hand side of (15) equals $-\mathbf{1}$.

We now define the self-adjoint matrices:

$$A = \sigma_x^1 \sigma_y^2, \quad B = \sigma_y^1 \sigma_x^2, \quad C = \sigma_x^1 \sigma_x^2, \quad D = \sigma_y^1 \sigma_y^2, \quad E = AB, \quad F = CD.$$

Using ii) and iii), we observe:

α) $[A, B] = 0$,

β) $[C, D] = 0$,

γ) $[E, F] = 0$.

The identity (15) can be rewritten as

$$EF = -\mathbf{1}. \tag{16}$$

But, using (9), α), β), γ), and (13), we get:

a) $v(EF) = v(E)v(F) = v(AB)v(CD)$

b) $v(AB) = v(A)v(B)$

c) $v(CD) = v(C)v(D)$

d) $v(A) = v(\sigma_x^1)v(\sigma_y^2)$

e) $v(B) = v(\sigma_y^1)v(\sigma_x^2)$

f) $v(C) = v(\sigma_x^1)v(\sigma_x^2)$

g) $v(D) = v(\sigma_y^1)v(\sigma_y^2)$

Combining (16) with (10) and a)–g), we get

$$v(EF) = -1 = v(\sigma_x^1)v(\sigma_y^2)v(\sigma_y^1)v(\sigma_x^2)v(\sigma_x^1)v(\sigma_x^2)v(\sigma_y^1)v(\sigma_y^2), \tag{17}$$

where the right-hand side equals $v(\sigma_x^1)^2 v(\sigma_y^2)^2 v(\sigma_y^1)^2 v(\sigma_x^2)^2$, since all the factors in the product appear twice.

But this last expression, being the square of a real number, is positive (actually, it is equal to 1, since the eigenvalues of the Pauli matrices in (17) are equal to -1 or $+1$), and so cannot equal -1. ■

A possible, but misleading, reaction to this theorem is to say that there is nothing new here, since it is well known that there is no quantum state that assigns a given value to all the spin variables in different directions simultaneously. But that misses the point: the theorem considers the possibility that there be other variables characterizing an individual system than its quantum state (in other words, that the quantum description is incomplete), variables whose values would be revealed by proper measurements. The theorem shows that, at least if the class of those variables is large enough, merely assuming the existence of those variables is impossible. Note that we are not assuming that there exists a theory about those “hidden” variables, telling us how they evolve in time for example, but merely that these variables exist and that their values agree with the quantum mechanical predictions.

The theorem means that the measuring devices play necessarily an active role and do not simply record pre-existing properties of the system. It is something Bohr always emphasized, for example in his discussion with Einstein in 1949, when he insisted on:

> [...] the *impossibility of any sharp distinction between the behavior of atomic objects and the interaction with the measuring instruments which serve to define the conditions under which the phenomena appear.*
>
> Niels Bohr [9, p. 210] (italics in the original)

But ordinary quantum mechanics does not tell us what this active role is, since there the measuring devices play the role of a *deux ex machina*.

2.3 A deeper problem: the need for an "ontology"

We now face a serious conundrum. There are two positions that might seem to justify the "no worry" attitude with respect to the meaning of the quantum state: *either* a proper quantum treatment of the measurement process would lead to a collapsed state and the need for two different laws of evolution would be eliminated, *or* the quantum state does not represent a single system but an ensemble of systems, each having its own individual properties that a measurement would simply reveal. But neither of these positions are defensible, either because the linearity of the Schrödinger equation leads necessarily to macroscopic superpositions, or because of the no hidden variables theorem.

Nevertheless (and it would be interesting to make a sociological study of this issue), it is probable that the creation of a definite quantum state by the interaction with a macroscopic apparatus, or the statistical interpretation, lies in the back of the mind of the physicists who are not bothered by the problems raised by quantum mechanics (who are of course the vast majority of physicists). It is unlikely that most physicists literally believe that the cat suddenly becomes alive or dead, simply because we look at it, especially if "looking" refers to the action of a mind independent of all physical laws[9].

[9] Indeed, if "looking" is a purely physical process, involving the eyes or the brain but obeying the linear laws of evolution of quantum mechanics, then one simply obtains more macroscopic superpositions, with a state of the eyes/brain seeing the live cat "plus" one seeing the dead cat, similar to (7) but with one more factor in each term, characterizing the state of eyes/brain:

$$c_1\varphi^{\uparrow}(z)|1\uparrow\rangle|\text{dead cat}\rangle|\text{the eyes/brain see the dead cat}\rangle$$
$$+c_2\varphi^{\downarrow}(z)|1\downarrow\rangle|\text{live cat}\rangle|\text{the eyes/brain see the live cat}\rangle.$$

However, one must distinguish those views that, if they were correct, would seem to justify the no worry attitude, from two other reactions to what has been described in the previous sections and that are associated with the "Copenhagen interpretation" and are the "official" or "orthodox" position (although, as we just said, it does not necessarily reflect what is in the minds of most physicists who adhere verbally to that position):

1) Claim that one cannot understand the microscopic world and that one must content oneself with predicting the results of measurements, which are necessarily macroscopic, and are thus described in a "classical" (i.e., understandable) language. For example, one could say that, if one prepares the particle in state (2), then, at the end of the experiment, the pointer of figures (1, 2) will point upwards for a fraction $|c_1|^2$ of cases, and it will point downwards for a fraction $|c_2|^2$ of cases. Then, one claims that this is all that one can say about the microscopic system, namely describe its interactions with measuring devices.

2) Introduce, by *decree*, a sharp distinction between the quantum and the classical world. One decides that, by definition, objects such as pointers or cats have well defined properties, while microscopic objects do not have any such properties.

   The problem with that approach is that it introduces a completely arbitrary separation: if an atom is governed by quantum laws, and if two atoms are also governed by those laws, what about ten atoms or a million atoms? Where does one put the dividing line?

   Let us stress that this not the same thing as a mathematical limit that one can get closer and closer to, because here we deal with a qualitative jump between "not having definite properties" and "having them".

We already explained why the first position is not satisfactory, since the whole of physics is then reduced to "predicting results of measurements" and one abandons the goal of describing the world outside the laboratories. Of course, it may be that one cannot do better. After all, why should mere human beings be able to understand the microscopic world? This may simply lie beyond our cognitive abilities. But, in order to accept that pessimistic conclusion, an argument must be given. And what one often hears, from the defenders of the "orthodoxy", is simply the claim that quantum mechanics is "complete" or that, since it works so well, no question should be asked. But those statements are assertions, not arguments.

In fact, a moment's reflection shows that quantum mechanics cannot be "complete". Indeed, let us go back to the example of Schrödinger's cat and consider one of the two wave functions: $|\text{live cat}\rangle$ or $|\text{dead cat}\rangle$.

These are not cats! A cat (or any other macroscopic object, like a pointer) is something located in space and that evolves in time. On the other hand, a wave function is a vector in a Hilbert space or, more concretely, a function defined on a high dimensional space, an element of $L^2(\mathbb{R}^N)$, where N equals three (i.e. the dimension of space) times the number of particles that the cat is composed of. Such a function simply does not have a value in the ordinary space $\mathbb{R}^3$!

One could say that the wave function represents all what we know about the cat, or that it contains all the information that we have about the cat. But, still, the cat, existing in three dimensional space, is a kind of object entirely different from a wave function.

So, even if one had spontaneous collapses of the wave function (and some modifications of Schrödinger's equation lead to such collapses, see [18, 4]), one would still need an "ontology" or, using a term invented by John Bell, "local beables". The word ontology, that may scare people as being too philosophical, simply refers to what exists, or, to be more precise, to what is postulated to exist by the physical theory. It could include atoms, elementary particles, stars, or fields. The word "beable" was invented in order to contrast it with the word "observable" that is so central to the standard quantum mechanical terminology and again refers to what exists, independently of whether we look at it or not. Finally beables are "local" if they are located in space, like cats or pointers[10].

But obviously we need to postulate the existence of some local beables, and even

[10] For a further discussion of the need of local beables in theories that modify Schrödinger's equation in order to produce spontaneous collapses, see [2]. The same problem occurs for the "many-worlds" theory of Everett, [17, 14], see [3].

ordinary quantum mechanics does that (without using that expression of course): the measuring devices are objects located in space[11]!

Of course, local beables are just another instance of what is called “hidden variables”; the fact that even ordinary quantum mechanics accepts the existence of some local beables shows that the word “hidden variables” is just a scare word that deters people from asking questions about the meaning of the quantum state.

The statistical interpretation discussed above does introduce local beables in some ways: it postulates that particles do exist and have definite properties, namely the values associated to the various observables being “measured”. But we have seen that this approach is inconsistent.

So, what we need is a theory that does introduce local beables or hidden variables, but in a way that is not contradicted by the no hidden variables theorem of section 2.2. We will sketch such a theory in the following sections.

3 The de Broglie–Bohm Theory

The de Broglie–Bohm theory that was elaborated first by de Broglie in 1924-1927 (see [13]) and rediscovered and developed by Bohm in 1952 [8] is:

- a “hidden variables theory”;
- that are not hidden;
- a theory that totally eliminate the role of the “observer”;
- a theory that is not refuted by the no hidden variables theorems such as the one of section 2.2; it is a statistical interpretation, but a consistent one;
- a theory that accounts for all the experiments justifying quantum mechanics;
- a theory that allows us to understand the active role of measuring devices, which Bohr emphasized, but without making it a philosophical a priori.

[11] We set aside here philosophical views such as solipsism or radical forms of idealism, for which there is nothing outside our consciousness, that physicists cannot adhere to if their goal is to understand the world.

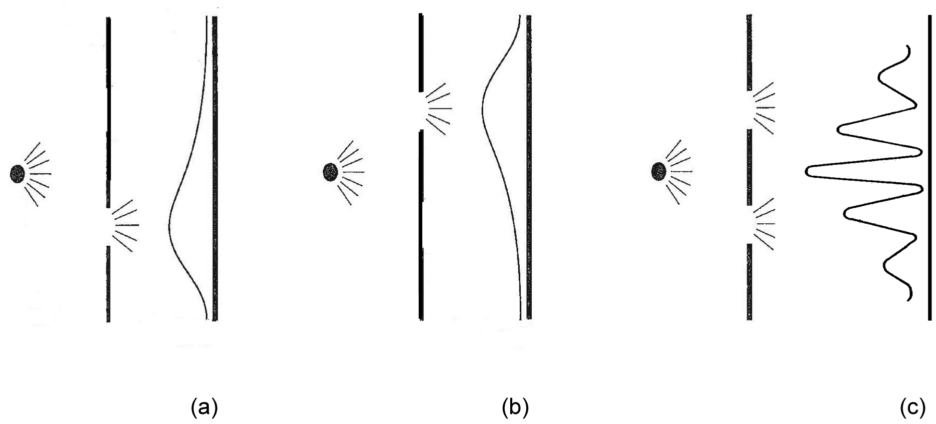

Fig. 5: The two slits experiment

3.1 The Equations of the de Broglie–Bohm Theory

Let us start with the well known double slit experiment: one sends electrons (one by one, so that they do not interact with each other) towards a wall in which two slits can be open or closed and behind which there is another wall, where one records the arrival of the particles. In figure 5, the density of electrons on the second wall is indicated, when one slit is open (a), or the other one (b) or both (c).

The usual way to describe this experiment is to say that, when only one slit is open, electrons behave like particles (parts (a) and (b) of figure 5), but when both slits are open, they behave like waves (part (c) of figure 5). But how can electrons be both waves and particles, and how can they "know" in advance whether one or two slits are open and thus how they must "behave"?

One could answer: elementary my dear Bohr! ! They are particles *guided* by waves.

In the de Broglie–Bohm theory, the complete state of a physical system is a pair $(|\text{quantum state}>, \mathbf{X})$, where $|\text{quantum state}>$ is the usual state, and $\mathbf{X} = (X_1, \ldots, X_N)$

represent the positions of the particules that exist, independently of whether one "looks" at them or one "measures" them (N here is the number of degrees of freedom of the system, so that each $X_i \in \mathbb{R}$).

These positions are the "hidden variables" (or the "local beables") of the theory, in the sense that they are not included in the purely quantum description $|$quantum state $>$, but they are not at all hidden: it is only the particles positions that one detects directly, for example on the screen in the two slit experiment.

The time evolution of the complete physical state is composed of two laws:

1. The usual evolution of the $|$ quantum state $>$, for all times, *whether one measures something or not*. If the state is only a wave function, $|$quantum state $>= \Psi(x_1, \dots, x_N)$, it obeys Schrödinger's equation:

$$i\partial_t \Psi = \mathcal{H}\Psi \tag{18}$$

where

$$\mathcal{H} = -\frac{1}{2}\Delta + V \tag{19}$$

is the quantum Hamiltonian ($\Delta = \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2}$), with, to simplify, $\hbar = 1$, and all the masses equal to 1.

2. The particle positions evolve in time $\mathbf{X} = \mathbf{X}(t)$ according to a guiding equation determined by the quantum state: their velocity is a function of the wave function. If one writes[12]:

$$\Psi(x_1, \dots, x_N) = R(x_1, \dots, x_N)e^{iS(x_1,\dots,x_N)},$$

then:

$$\frac{dX_k(t)}{dt} = \nabla_k S(X_1(t), \dots, X_N(t)). \tag{20}$$

[12] We use lower case letters for the generic arguments of the wave function and upper case ones for the actual positions of the particles.

Or, more generally, for quantum states that are multi-component wave functions:

$$\frac{dX_k(t)}{dt} = V_\Psi^k(\mathbf{X}(t)) = \quad \frac{\mathrm{Im}(\Psi^* \cdot \nabla_k \Psi)}{\Psi^* \cdot \Psi}(X_1(t), \ldots, X_N(t)). \tag{21}$$

where $\cdot$ stands for the scalar product between the components of the quantum state.

The origin of this equation is not mysterious; it is of the form:

$$\frac{d\mathbf{X}}{dt} = \frac{\mathbf{J}}{\rho} \tag{22}$$

where $\mathbf{J} = \mathrm{Im}(\Psi^* \cdot \nabla\Psi)$ is the current associated with the "conservation of probability" $\rho = |\Psi|^2$:

$$\frac{\partial\rho}{\partial t} + \nabla \cdot \mathbf{J} = 0. \tag{23}$$

Bohm rewrote this equation as a second order equation (in time), namely as a Newton's equation with a modified potential: $V_{class} \rightarrow V_{class} + V_Q(\Psi)$ where $V_Q(\Psi) = \frac{-\Delta|\Psi|}{2|\Psi|}$. If one takes into account $V_Q(\Psi)$, one may reason more or less "classically".

Let us come back to the two slits experiment: in figure 6, we show a numerical solution of the de Broglie-Bohm dynamics in that situation.

Note that the motion *in vacuum* behind the slits is highly *non classical* ! Newton's first law (rectilinear motion in the absence of forces) is not satisfied. If one reasons, as Bohm did, in terms of potentials, behind the slits $V_{class} = 0$, but $V_Q(\Psi) \neq 0$, since V_Q depends on the wave function that propagates behind the slits.

Note also that one can determine a posteriori through which slit the particle went! Indeed, there is a nodal line that the particles cannot cross: since there is a symmetry between the top and the bottom of the image, the gradient of Ψ is tangent to the line separating those two parts of the image; thus, because of (20), the velocity of the particle is also tangent to that line and the particles cannot go through it.

It is interesting to compare this numerical solution to an experiment published in *Science* in June 2011 [22], see figure 7, and that shows trajectories of photons obtained though a series of "weak measurements", that are indirect measurements. The profile of those trajectories is very similar to the one of figure 6.

In a rather standard quantum mechanical textbook, one reads:

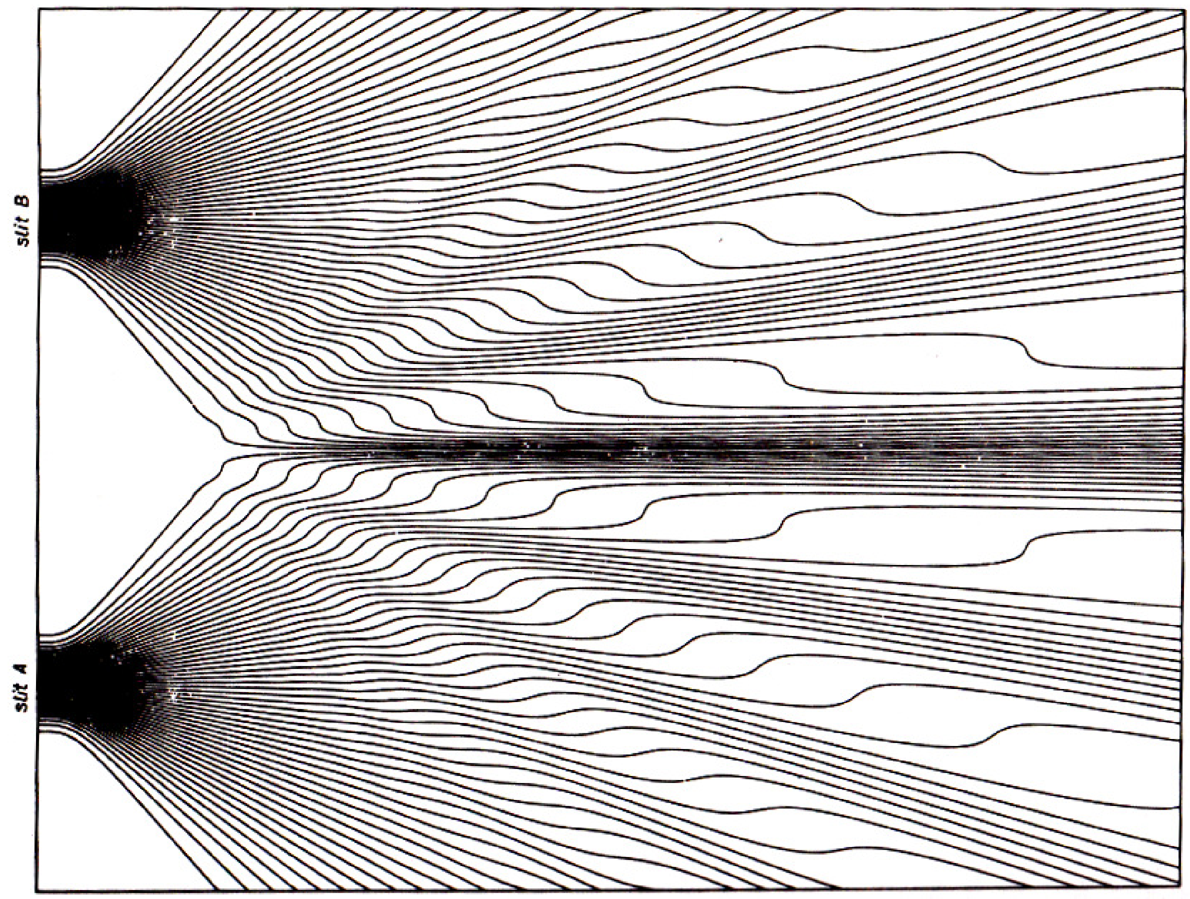

Fig. 6: Trajectories of the de Broglie-Bohm dynamics in the two slits experiment [29]

> It is clear that [the results of the double-slit experiment] can in no way be reconciled with the idea that electrons move in paths. [. . .] In quantum mechanics there is no such concept as the path of a particle.
>
> Lev Landau and Evgeny Lifshitz [23, p. 2]

But is it so clear, given what we just saw? Bell expressed very clearly the natural aspect of the de Broglie-Bohm theory in the double slit experiment:

> Is it not clear from the smallness of the scintillation on the screen that we have to do with a particle? And is it not clear, from the diffraction and interference patterns, that the motion of the particle is directed by a wave? De Broglie showed in detail how the motion of a particle, passing through just one of

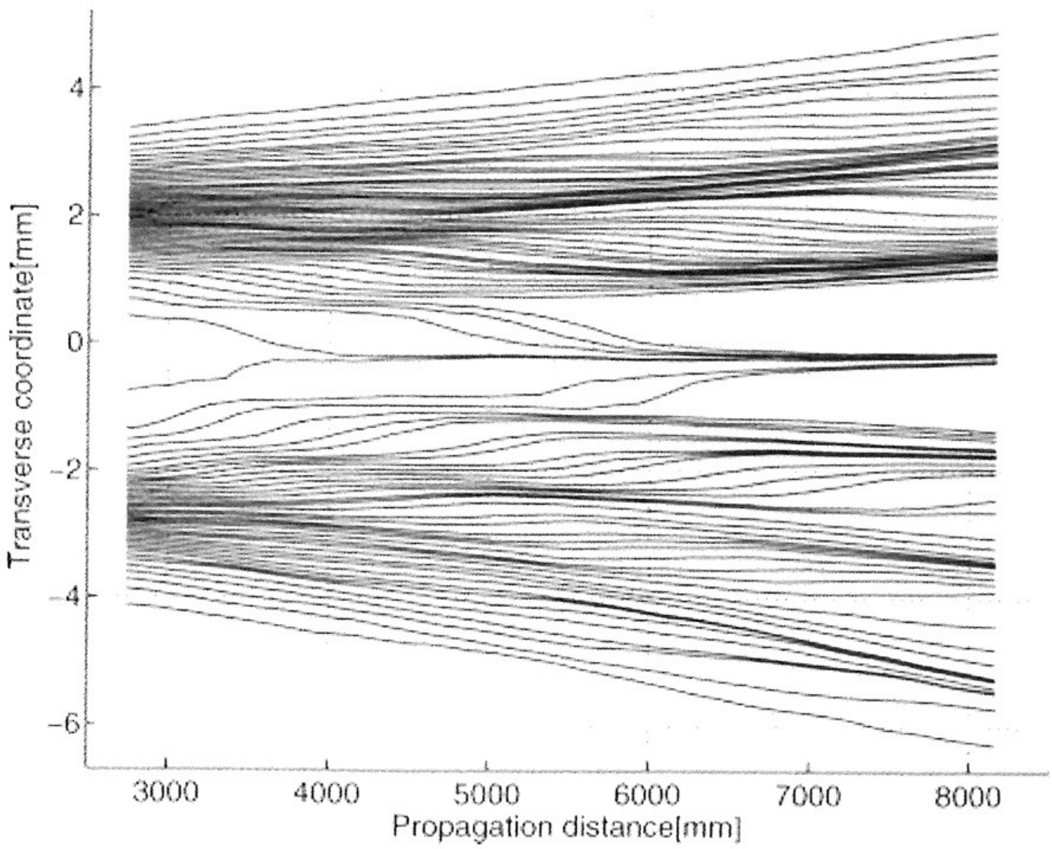

Fig. 7: Weak measurement of photon trajectories in the double slit experiment due to [22]

> two holes in the screen, could be influenced by waves propagating through both holes. And so influenced that the particle does not go where the waves cancel out, but is attracted to where they cooperate. This idea seems to me so natural and simple, to resolve the wave–particle dilemma in such a clear and ordinary way, that it is a great mystery to me that it was so generally ignored.
>
> John Bell [7, p. 191]

In order to understand why the de Broglie-Bohm theory reproduces the usual quantum predictions, one must use a fundamental consequence of that dynamics: *equivariance*, that is illustrated by figure 8, where each line describes a possible trajectory (for simplicity, we consider a one-dimensional system).

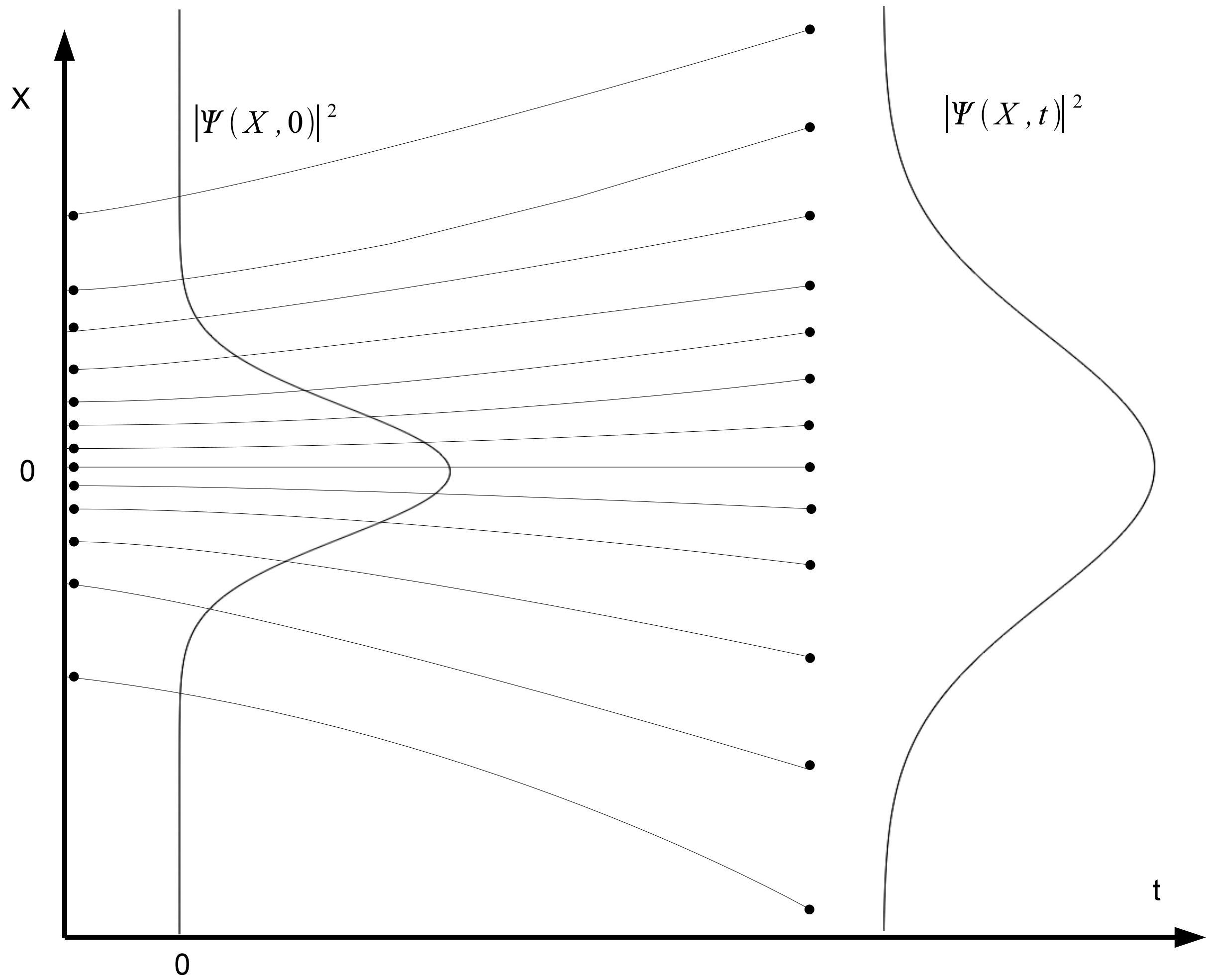

Fig. 8: Illustration of the equivariance of the de Broglie-Bohm dynamics

If the initial distribution of particles is given by a density ρ_0, the density at a later time will be given by ρ_t:

$$\rho_0(X) = |\Psi(X,0)|^2 \to \rho_t(X) = |\Psi(X,t)|^2, \tag{24}$$

where $\Psi(X,t)$ comes from Schrödinger's equation (18), and $\rho_t(X)$ comes from the pilot wave equation (20)[13]. This follows easily from equations (22, 23) (See [10] Appendix 5.C for more details).

In figure 8 one represents a random distribution of initial positions whose density is

[13] For any deterministic dynamical system where an initial condition X_0 leads to a state X_t at time t, one may define the evolution of a density of particles by the formula: $\rho_t(X) = \rho_0(X_0(X,t))$ where $X_0(X,t)$ is the initial condition at time 0 that leads to the state $X_t = X$ at time t.

approximately given by $|\Psi(X,0)|^2$ (where the variable X is on the vertical axis) and one sees that at a later time t this density will approximately be given by $|\Psi(X,t)|^2$, each line representing the result of the de Broglie-Bohm dynamics, for a given initial condition on the vertical axis.

Because of equivariance, the quantum predictions for the position measurements are verified if one assumes that the initial density of particles satisfy $\rho_0(X) = |\Psi(X,0)|^2$. This hypothesis on initial distributions is called *quantum equilibrium*; its justification is too long to be discussed here (see [15] or [10] section 5.1.7).

Thus, in the de Broglie-Bohm theory, the quantum state Ψ has a double status:

– It generates, though equations (20) or (21) the motion of particles.

– It also governs the statistical distribution of the particles positions, given by $|\Psi|^2$.

One can compare the quantum state with the Hamiltonian in classical physics, since the latter generates the motion of particles, through Hamilton's equation, and also gives the statistical distribution at equilibrium. Formally, the analogy is as follows: $\mathcal{H} \sim -\log\Psi$ and $|\Psi|^2 \sim \exp(-\beta\mathcal{H})$, with $\beta = 2$.

In the de Broglie-Bohm theory, the only "hidden variables" that are introduced are the particle positions. That is why that theory is not refuted by the theorems on the impossibility of hidden variables that assume the existence of a function v giving pre-existing values to all "observables" belonging to a certain set, for example the spin values in different directions or both positions and momenta[14]. There is no "no hidden variables" theorem against giving pre-existing values only to positions, as is done in the de Broglie-Bohm theory.

The fact that the only hidden variables introduced in the de Broglie-Bohm theory are the particles' positions, and therefore their trajectories, is sometimes regarded as an argument against that theory: Why focus so rigidly on positions and not also on other observables? But a moment's reflection shows that the situation is completely similar in classical mechanics: all quantities like the velocities, energies, angular momenta, are functions of the trajectories of the particles. We shall see that, in quantum mechanics, the values of all "observables" other than position can be derived from an analysis of the

[14] See [12] and [10, p.43] for this version of the no hidden variables theorem.

particles' trajectories[15].

Of course, we still need to understand how the measurements of observables other than position work. Let us start with the spin[16] and then consider the momentum.

3.2 The "measurement" of spin in the de Broglie-Bohm theory

A consequence of the theorems on the non existence of hidden variables of section 2.2 is that what one calls a measurement – other than measurements of position – do not measure any pre-existing property of the particle. In order to understand that in the de Broglie-Bohm theory, consider the measurement of the spin in a state which is a superposition of $|1\uparrow>$ and $|1\downarrow>$, that we describe in an idealized form:

$$\Psi(z)(|1\uparrow> + |1\downarrow>) \tag{25}$$

z being the vertical direction (see figure 9). We will assume that the spatial part of the state, namely the wave function $\Psi(z)$ is symmetrical: $\Psi(z) = \Psi(-z)$. That implies that the line $z = 0$ is a nodal line, through which the gradient of $\Psi(z)$ is zero and that the particles cannot cross, by (20) (the situation is similar to the one of figure 6). Ψ is also a function of the horizontal variables x (the particle moves rightwards in the x direction), but we will leave aside that variable.

In figure 9, H denotes a magnetic field; the disks represent (in a very idealized way) the support of the spatial part of the wave function. The part $|1\uparrow>$ of the quantum state always goes in the direction of the field (which gives rise to the state $\Psi(z-t)|1\uparrow>$) and the part $|1\downarrow>$ always goes in the direction opposite to the field ($\Psi(z+t)|1\downarrow>$). But the particle, it it starts initially in the part of the wave function above the nodal line $z = 0$ (produced by the symmetry of the wave function), will always go up, because it cannot cross that nodal line[17].

[15] We thank Sheldon Goldstein for this remark.

[16] The next section is based on Chapter 7 of David Albert's book "Quantum Mechanics and Experience" [1].

[17] Since the particles here have spin, the guiding equation is (21), not (20), but this does not affect our qualitative discussion.

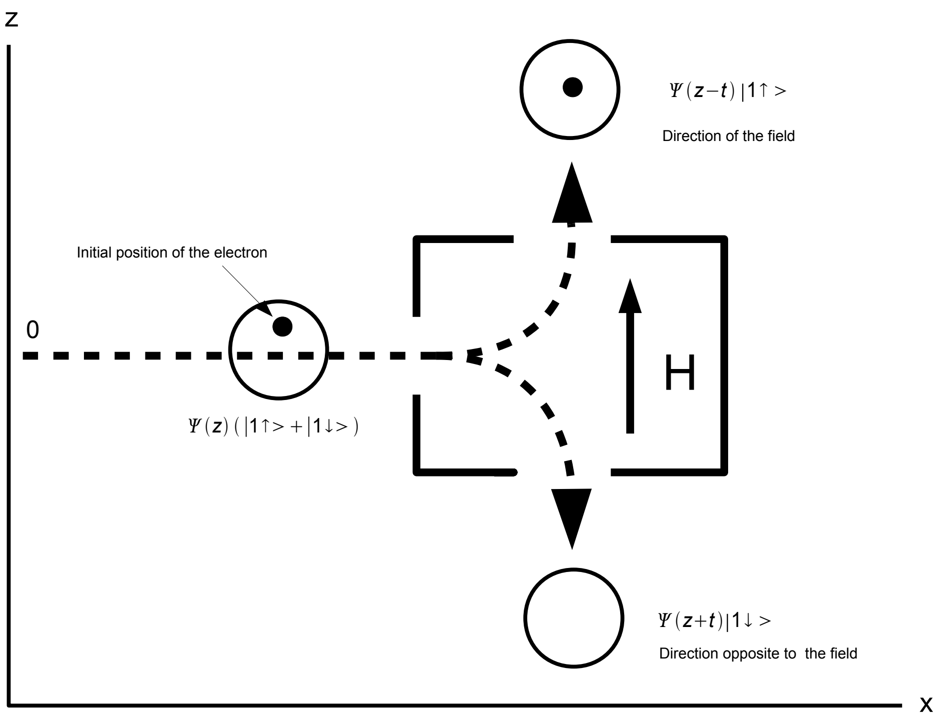

Fig. 9: An idealized spin measurement

Thus, if one reverses the direction of the field, as in figure 10, the particle will again go upwards. But then, what was "spin up" becomes "spin down" (i.e. going in the direction opposite to the one of the field), although one "measures" the same "observable", i.e. the spin in the *same* direction in both set-ups, and with the *same initial conditions* for the particle (both its wave function and position) but with different arrangements of the apparatus.

Therefore, the measuring device is not "passive" (it does not record any intrinsic property of the particle pre-existing to the measurement) but "active". This justifies Bohr's intuition, but by incorporating it in the theory itself, not as a *deus ex machina.*

Using equivariance, one can show that the statistical predictions for the results of "spin measurements" in the de Broglie-Bohm theory do not depend on the choice of the orientation of the field and coincide with the usual ones.

Note also that both parts of the wave function in figures 9 and 10 continue to evolve

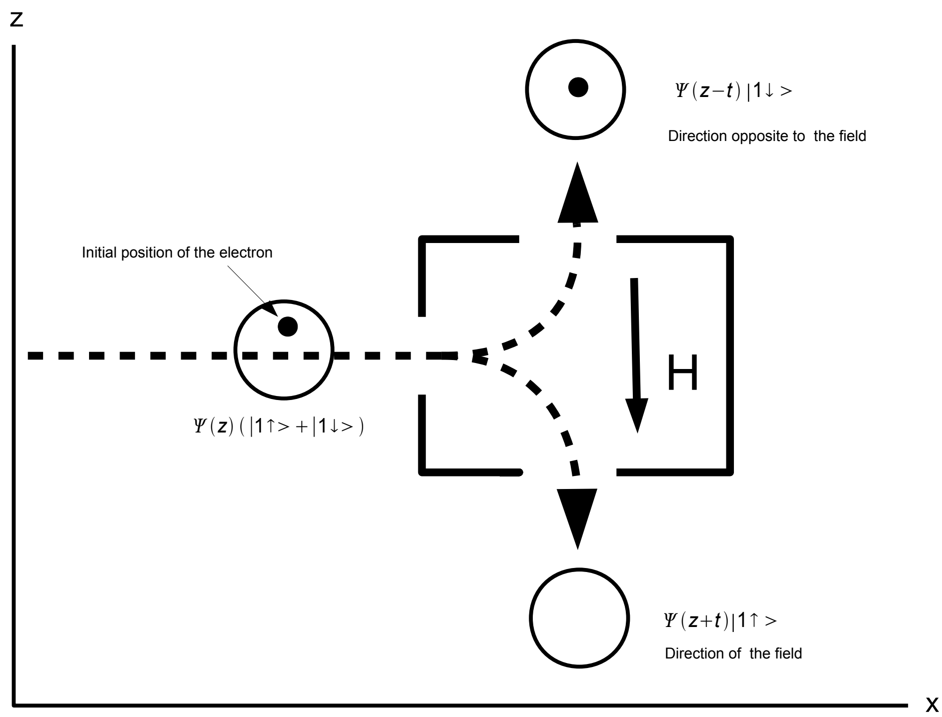

Fig. 10: An idealized spin measurement with the direction of the field reversed with respect to the one of figure 9

according to the usual equations. But the particle is guided only by the part of the wave function is the support of which it is. Which means that one can, in practice and in certain cases, reduce the wave function and only keep the part in the support of which the particle is. In figures 9 and 10, it could happen that those two parts of the wave function recombine later and therefore one cannot forget one of those parts (the one in the support of which the particle does not find itself).

But one can show that, when the particle interacts with a macroscopic apparatus and that one obtains a state like (6), then the recombination of the two wave functions is in practice impossible[18] and one may, again in practice, keep only the part of the wave func-

[18] As mentioned in section 2.1, this is called decoherence. It is important to underline that, if this notion is essential to understand why there is in practice a reduction of the quantum state in the de Broglie-Bohm theory, it is not sufficient, by itself, since nothing distinguishes, in the usual formalism, the two parts of the wave function, even if the latter "decohere". In the de Broglie-Bohm theory, the two

tion in the support of which the particle is if one wants to analyze its future behavior. In that sense, one "reduces" the quantum state and this operation is "emergent", somewhat like the irreversibility of macroscopic laws in classical physics[19].

3.3 The "measurement" of momentum in the de Broglie–Bohm theory

What about the "momentum measurements" ? If, in the de Broglie-Bohm theory, particles have trajectories, they must also have a velocity at all times. But isn't having both a position and a velocity at the same time contradicted by Heisenberg's inequalities[20]?

Not at all! To understand what is going on, consider a simple example, namely a particle in one space dimension with initial wave function: $\Psi(x,0) = \pi^{-1/4}\exp(-x^2/2)$. Since this function is real, its phase $S = 0$ and all particles are at rest (by equation (20): $\frac{dX(t)}{dt} = \frac{\partial S(X(t),t)}{\partial x}$).

Nevertheless, the measurement of momentum p must have, according to the usual quantum predictions, a probability distribution whose density is given by the square of the Fourier transform of $\Psi(x,0)$, i.e. by $|\hat{\Psi}(p)|^2 = \pi^{-1/2}\exp(-p^2)$.

Isn't there a contradiction here?

In order to answer that question, one must first see *how* one measures that momentum. One way to do it is to let the particle evolve and to detect its asymptotic position $X(t)$ as $t \to \infty$. Then, one has $p = \lim_{t\to\infty} \frac{X(t)}{t}$ (setting the mass $m = 1$).

Consider the free evolution of the initial wave function $\Psi(x,0) = \pi^{-1/4}\exp(-x^2/2)$. The solution of Schrödinger's equation ((18), with $V = 0$) with that initial condition is:

parts differ because the particle is in the support of only one of them.

[19] In fact, one can introduce a notion of wave function for a subsystem (of the Universe) that coincides with the wave function used in quantum mechanics and that does collapse when collapses occur according to the standard approach, see [20] for a detailed discussion. The wave function that never collapses is the one of the Universe.

[20] There exists also a no hidden variables theorem, due to Robert Clifton [12], preventing us from assigning both a position and a velocity to two particles on a line, in such a way that their statistical distribution coincides with the usual quantum predictions. See [10, p. 43] or [11] for a discussion of that theorem.

$$\Psi(x,t) = \frac{1}{(1+it)^{1/2}}\frac{1}{\pi^{1/4}}\exp\left[-\frac{x^2}{2(1+it)}\right], \tag{26}$$

and thus

$$|\Psi(x,t)|^2 = \frac{1}{\sqrt{\pi[1+t^2]}}\exp\left[-\frac{x^2}{1+t^2}\right]. \tag{27}$$

If one writes $\Psi(x,t) = R(x,t)\exp[iS(x,t)]$, one gets (up to a constant in x):

$$S(x,t) = \frac{tx^2}{2(1+t^2)},$$

and the guiding equation (20) becomes:

$$\frac{d}{dt}X(t) = \frac{tX(t)}{1+t^2}, \tag{28}$$

whose solution is:

$$X(t) = X(0)\sqrt{1+t^2}. \tag{29}$$

This gives the explicit dependence of the position of the particle as a function of time. If the particule is initially at $X(0) = 0$, it does not move; otherwise, it moves asymptotically, when $t \to \infty$, as $X(t) \sim X(0)t$.

The probability density of $X(t) = x$ is given by $|\Psi(x,t)|^2$ (because of equivariance)[21]. Thus, the probability that the result of a "measurement" of the momentum belongs to a set $A \subset \mathbb{R}$ is $\lim_{t\to\infty}\int_{At}|\Psi(x,t)|^2 dx$.

Changing variables $x = pt$, one gets:

$$\int_{At}|\Psi(x,t)|^2 dx = t\int_A |\Psi(pt,t)|^2 dp \tag{30}$$

and, by (27), one obtains:

$$t\int_A |\Psi(pt,t)|^2 dp = t\int_A \frac{1}{\sqrt{\pi(1+t^2)}}\exp\left(-\frac{p^2t^2}{1+t^2}\right)dp, \tag{31}$$

whose limit, when $t \to \infty$, is

$$\int_A \pi^{-1/2}\exp(-p^2)dp = \int_A |\widehat{\Psi}(p,0)|^2 dp.$$

This is the quantum prediction! But it does *not* measure the initial velocity (which is zero for all the particles).

[21] One can also check that explicitly using (27) and (29).

Thus, the particles do have a trajectory and, at all times, *a position and a velocity.*

But, if one understands how “measurements” work, there is no contradiction between the de Broglie-Bohm theory and the quantum predictions and, in particular, Heisenberg’s inequalities. The latter are simply relations between variances of results of measurements, but they imply nothing whatsoever on what exists or does not exists outside of measurements, since those inequalities are simply mathematical consequences of the quantum formalism and that formalism says nothing about the world outside of measurements.

The lesson to be drawn from all this is that any measurement other than a position measurement is an *interaction* between the microscopic physical system and the macroscopic measuring device, or, to use a fashionable expression, the values of all observables other than positions are *emergent*, but it is only in the de Broglie-Bohm theory that one can understand how that interaction works.

4 Conclusions

People often wonder what is the relation between the de Broglie-Bohm theory and ordinary quantum mechanics? A brief answer is that those are the same theories!

But then, what is the de Broglie-Bohm theory good for?

A more precise and correct answer is that one theory (the de Broglie- Bohm one) is a theory about the World, the other (ordinary quantum mechanics) is not. It does not even pretend to a theory about the World, but only an algorithm allowing us to predict results of measurements!

Ordinary quantum mechanics is simply a truncated version of the de Broglie-Bohm theory: one forgets the de Broglie-Bohm trajectories, this does not affect the empirical predictions of ordinary quantum mechanics (which are actually a consequence of the de Broglie-Bohm theory), but “only” creates entire libraries full of confusions, mysticism and bad philosophy.

One could say that the main virtue of the Broglie-Bohm theory is to clarify our concepts, but that is no small achievement, given what has been said both by scientists and non-scientists about the centrality of the observer or the disappearance of objective reality in quantum mechanics.

John Bell explained in detail what the Broglie-Bohm theory is good for:

> Why this necessity to refer to 'apparatus' when we would discuss quantum phenomena? The physicists who first came upon such phenomena found them so bizarre that they despaired of describing them in terms of ordinary concepts like space and time, position and velocity. The founding fathers of quantum theory decided even that no concepts could possibly be found which could permit direct description of the quantum world. So the theory which they established aimed only to describe systematically the response of the apparatus. And what more, after all, is needed for applications? [. . .]
>
> The 'Problem' then is this: how exactly is the world to be divided into a speakable apparatus . . . that we can talk about . . . and unspeakable quantum system that we can not talk about? How many electrons, or atoms, or molecules, make an 'apparatus'? The mathematics of the ordinary theory require such a division, but says nothing about how it is to be made. In practice the question is resolved by pragmatic recipes which have stood the test of time. But should not fundamental theory permit exact mathematical formulation?
>
> Now in my opinion the founding fathers were in fact wrong on this point. The quantum phenomena do *not* exclude a uniform description of micro and macro worlds . . . system and apparatus. It is *not* essential to introduce a vague division of the world of this kind. This was indicated already in 1926 by de Broglie, when he answered the conundrum
>
> wave or particle ?
>
> by
>
> wave *and* particle.
>
> But by the time this was fully clarified by Bohm in 1952, few theoretical physicists wanted to hear about it. The orthodox line seemed fully justified by practical success. Even now the de Broglie-Bohm picture is generally ignored,

> and not taught to students. I think this is a great loss. For that picture exercises the mind in a very salutary way.
>
> The de Broglie-Bohm picture disposes of the necessity to divide the world somehow into system and apparatus.
>
> John Bell [7, 170-171]

Bell was also wondering:

> Why is the pilot wave picture ignored in textbooks? Should it not be taught, not as the only way, but as an antidote to the prevailing complacency? To show that vagueness, subjectivity, and indeterminism are not forced on us by experimental facts, but by deliberate theoretical choice?
>
> John Bell [7, p. 160]

Let us end by quoting an ex-physics student, whose sentiments are close to mine when I was a student:

> My interest has always been to understand what the world is like. This is the main reason that I majored in physics: if physics is the study of nature, then to understand nature one should learn physics first. But my hopes were disappointed by what is (or at least seems to be) commonly accepted in many physics departments all over the world: after quantum mechanics, we should give up the idea that physics provides us with a picture of reality. At first, I believed this was really the case and I was so disappointed that I decided to forget about my "romantic" dream.
>
> At some point, [...] I realized that some of the things I took for granted were not so obviously true, and I started to regain hope that quantum mechanics was not really the "end of physics" as I meant it. Therefore, I decided to go to

> graduate school in physics to figure out what the situation really was. While taking my PhD in the foundations of quantum mechanics, I understood that what physicists thought was an unavoidable truth was instead a blunt mistake: quantum mechanics does not force us to give up anything, and certainly not the possibility to investigate reality through physics.
>
> Valia Allori[22]

One may also quote the shortest judgment on “Copenhagen”:

> A philosophical extravaganza dictated by despair.
>
> Erwin Schrödinger[23]

Acknowledments

I thank Sheldon Goldstein for interesting discussions on the topics discussed in this paper.

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