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Mathematical Statistical Physics: Assignment 1

Problem 1: *In high dimension, oranges are mostly peel.* (hand in, 25 points)
Show that for all $\varepsilon, \delta \in (0,1)$ there is $d_0 \in \mathbb{N}$ such that, for all $d > d_0$, a fraction of at least $1-\varepsilon$ of the volume of the unit ball in $\mathbb{R}^d$ is contained in the shell of thickness δ underneath the surface.

Problem 2: *Normalization of the Gaussian* (don't hand in)
Show that for all $\mu \in \mathbb{R}$ and $\sigma > 0$,

$$\int_{-\infty}^{+\infty} dx \, \frac{1}{\sqrt{2\pi}\sigma} \, \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) = 1\,.$$

Problem 3: *Gamma function* (hand in, 15 points)
Show that the Gamma function, defined on $(0,\infty)$ by $\Gamma(\alpha) = \int_0^\infty dt\, t^{\alpha-1} e^{-t}$, has the following properties.

(a) $\Gamma(\alpha+1) = \alpha\,\Gamma(\alpha)$.

(b) $\Gamma(1) = 1$. Thus, $\Gamma(n) = (n-1)!$ for $n \in \mathbb{N}$.

(c) $\Gamma(1/2) = \sqrt{\pi}$ (Hint: substitute $s = \sqrt{t}$). Thus, $\Gamma(n+1/2) = \dfrac{(2n)!\sqrt{\pi}}{4^n n!}$.

Problem 4: *Spherical coordinates in* $\mathbb{R}^d$ (hand in, 30 points)
They are defined by

$$\begin{aligned}
x_1 &= r\cos\phi_1 \\
x_2 &= r\sin\phi_1\cos\phi_2 \\
x_3 &= r\sin\phi_1\sin\phi_2\cos\phi_3 \\
&\cdots \\
x_{d-1} &= r\sin\phi_1\cdots\sin\phi_{d-2}\cos\phi_{d-1} \\
x_d &= r\sin\phi_1\cdots\sin\phi_{d-1}
\end{aligned} \tag{1}$$

with $r \in [0,\infty)$, $\phi_1,\ldots,\phi_{d-2} \in [0,\pi]$, and $\phi_{d-1} \in [0,2\pi)$.

(a) Show that for fixed $r > 0$, the image of the ϕ coordinates is the sphere of radius r, $\mathbb{S}_r^{d-1} = \big\{(x_1,\ldots,x_d) \in \mathbb{R}^d : x_1^2 + \ldots + x_d^2 = r^2\big\}$.

(b) Show that the Jacobian determinant of the coordinate transformation (1) is

$$J = r^{d-1} \sin^{d-2}\phi_1 \sin^{d-3}\phi_2 \cdots \sin\phi_{d-2}\,.$$

(In other words, the ($d-1$-dimensional) area dA of a surface element is

$$dA = r^{d-1} \sin^{d-2}\phi_1 \sin^{d-3}\phi_2 \cdots \sin\phi_{d-2}\, d\phi_1\, d\phi_2 \cdots d\phi_{d-1}\,,$$

and the (d-dimensional) volume of a volume element is $dV = dr\, dA$.)

(c) Show that the area of $\mathbb{S}_r^{d-1}$ is given by

$$A = \frac{2\pi^{d/2}}{\Gamma(d/2)} r^{d-1}\,, \tag{2}$$

where Γ is the Gamma function, and the volume of the ball $B_r \subset \mathbb{R}^d$ by

$$V = \frac{\pi^{d/2}}{\Gamma(1+d/2)} r^{d}\,. \tag{3}$$

Hint: Use without proof that $\displaystyle\int_0^\pi d\phi\, \sin^k\phi = \sqrt{\pi}\,\Gamma\big(\tfrac{k+1}{2}\big)/\Gamma\big(\tfrac{k+2}{2}\big)$.

Problem 5: *Belt theorem* (hand in, 30 points)
Show that for every $\varepsilon, \delta \in (0,1)$ there is $d_0 \in \mathbb{N}$ such that, for all $d > d_0$, a fraction of at least $1-\varepsilon$ of the surface area of the unit sphere in $\mathbb{R}^d$ lies within a belt of width 2δ around the equator, $\{\boldsymbol{x} \in \mathbb{R}^d : -\delta < x_1 < \delta, |\boldsymbol{x}| = 1\}$. ("In high dimension, most points on the sphere are near the equator.")

Hint: Express the area of a surface of revolution as an integral along the axis of revolution, then generalize to higher dimension. The resulting integral is hard to solve explicitly, but we only need to show that the complementary integral (the surface outside the belt) is small. You may use without proof that there is $C > 0$ such that for all $n \in \mathbb{N}$,

$$\frac{\Gamma(n)}{\Gamma(n-1/2)} \le \frac{\Gamma(n+1/2)}{\Gamma(n)} \le C\sqrt{n}\,. \tag{4}$$

Problem 6: *Non-global solution* (don't hand in)
Verify that the trajectory (2.10) in the lecture notes is a solution of the equation of motion (2.1).

Problem 7: *Variance of a random variable* (don't hand in)
Let $\mathbb{E}X$ denote the expectation value of the random variable X. The variance of X is defined as $\operatorname{Var} X = \mathbb{E}\big[(X-\mathbb{E}X)^2\big]$. Show that $\operatorname{Var} X = \mathbb{E}(X^2) - (\mathbb{E}X)^2$.

Hand in: Wednesday, April 24, 2019 in the exercise class