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Summer semester 2019
April 24, 2019

Math Stat Phys: In-Class Problem Sheet 1

Problem 1: *Probability densities*
Let $D = \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq y \leq 1\}$. The random variables X, Y have joint distribution density $\rho(x, y) = \mathscr{N}\, 1_D(x, y)$, where 1_D denotes the characteristic function (indicator function) of the set D.

(a) Determine the normalization constant $\mathscr{N}$.

(b) Determine the marginal distribution density $\rho_X(x)$ of the random variable X.

(c) Let $Z = X^2$. Determine the distribution density $\rho_Z(z)$ of the random variable Z.

(d) Determine $\mathbb{E}Z$ in two ways: First by computing $\int dz\, z\, \rho_Z(z)$, and second by computing $\int dx\, x^2\, \rho_X(x)$.

Problem 2: *Convex sets and functions*
A subset K of the real vector space V is called *convex* iff for any two points $x, y \in K$ the line segment connecting them lies in K, i.e., iff $\forall t \in [0, 1] : \; tx + (1 - t)y \in K$.

(a) Show: If $K \subseteq V$ is convex, and if $x_1, \ldots, x_k \in K$ and $\alpha_1, \ldots, \alpha_k \geq 0$ with $\alpha_1 + \ldots + \alpha_k = 1$, then $\sum_{i=1}^k \alpha_i x_i \in K$. Such a linear combination is called a *convex combination*.

(b) For any subset $M \subseteq V$ let $H(M)$ be the intersection of all convex sets containing M as a subset. Show that $H(M)$ is convex. It is called the *convex hull* of M.

(c) Show that $H(M)$ is the set of all convex combinations of elements of M.

Let $G \subseteq V$. A function $f : G \to \mathbb{R}$ is called *convex* iff the set M above its graph, $M = \{(x, y) \in G \times \mathbb{R} | y \geq f(x)\}$, is convex.

(d) Show that convexity of G is necessary for convexity of f.

(e) Show that f is convex iff for all $x, y \in G$, $f_{xy}(t) := f\big(tx + (1 - t)y\big)$ defines a convex function $f_{xy} : [0, 1] \to \mathbb{R}$.