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Mathematical Statistical Physics: Assignment 2

Problem 8: *Conservation laws* (hand in, 40 points)
Prove Proposition 1, i.e., the conservation laws of energy, momentum, and angular momentum, assuming the equation of motion (2.1).

Problem 9: *Elastic collision of two billiard balls in* $\mathbb{R}^3$ (don't hand in)
Prior to the collision, the two balls of radius $a > 0$ and mass $m > 0$ move at constant velocities. Let the locations at the time of collision be $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$ with $|\boldsymbol{q}_1 - \boldsymbol{q}_2| = 2a$, the momenta prior to the collision $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ with $(\boldsymbol{p}_2 - \boldsymbol{p}_1) \cdot (\boldsymbol{q}_2 - \boldsymbol{q}_1) < 0$ (they move towards each other, not away), and let $\boldsymbol{\omega}$ be the unit vector from ball 1 to ball 2, $\boldsymbol{\omega} = (\boldsymbol{q}_2 - \boldsymbol{q}_1)/2a$. Assume that the balls cannot spin. Show that conservation of energy, momentum, and angular momentum allow only two possibilities for the momenta $\boldsymbol{p}_1'$ and $\boldsymbol{p}_2'$ after the collision: Either $\boldsymbol{p}_1' = \boldsymbol{p}_1$ and $\boldsymbol{p}_2' = \boldsymbol{p}_2$ (which is impossible if the balls cannot pass through each other), or the $\boldsymbol{\omega}$ components of the $\boldsymbol{p}_k$ get exchanged while the components perpendicular to $\boldsymbol{\omega}$ remain unchanged. Show further that the latter case means

$$\boldsymbol{p}_1' = \boldsymbol{p}_1 - [(\boldsymbol{p}_1 - \boldsymbol{p}_2) \cdot \boldsymbol{\omega}]\boldsymbol{\omega}\,, \quad \boldsymbol{p}_2' = \boldsymbol{p}_2 + [(\boldsymbol{p}_1 - \boldsymbol{p}_2) \cdot \boldsymbol{\omega}]\boldsymbol{\omega}\,.$$

("Elastic" means conserving energy. Here, $E = \boldsymbol{p}_1^2/2m + \boldsymbol{p}_2^2/2m$ and $\boldsymbol{L} = \boldsymbol{q}_1 \times \boldsymbol{p}_1 + \boldsymbol{q}_2 \times \boldsymbol{p}_2$.)

Problem 10: *Normalizing factor of the Gaussian in d dimensions* (hand in, 20 points)
Determine the normalizing factor $\mathscr{N}$ in the expression

$$\rho(\boldsymbol{x}) = \mathscr{N}\, \exp\bigl(-\tfrac{1}{2}(\boldsymbol{x} - \boldsymbol{\mu})^T C^{-1} (\boldsymbol{x} - \boldsymbol{\mu})\bigr)$$

for the density of $\mathcal{N}^d(\boldsymbol{\mu}, C)$, the Gaussian distribution in $\mathbb{R}^d$ with mean $\boldsymbol{\mu}$ and covariance matrix C. You can use the answer for $d = 1$ without proof.

Problem 11: *Distance of measures* (hand in, 20 points)
The *total variation distance* of two finite measures μ, ν on the same σ-algebra $\mathscr{A}$ is defined by

$$d(\mu, \nu) = \sup_{A \in \mathscr{A}} \bigl(\mu(A) - \nu(A)\bigr) + \sup_{A \in \mathscr{A}} \bigl(\nu(A) - \mu(A)\bigr)\,.$$

Now let μ, ν be probability measures.
(a) (10 points) Show that $d(\mu, \nu) = 2 \sup_{A \in \mathscr{A}} \bigl|\mu(A) - \nu(A)\bigr|$.
(*Hint*: $\nu(A) - \mu(A) = \mu(A^c) - \nu(A^c)$.)
(b) (10 points) Show that if μ and ν possess densities f and g relative to the measure λ on $\mathscr{A}$, then the total variation distance coincides with the L^1 norm of $f - g$,

$$d(\mu, \nu) = \int \lambda(dx)\, |f(x) - g(x)|\,.$$

Problem 12: *Speeds of molecules* (hand in, 20 points)
(a) (10 points) Determine the most probable value $v_{\max}$ of the speed $v = |\boldsymbol{v}|$ according to the Maxwellian distribution

$$\rho(\boldsymbol{v}) = \mathscr{N}\ \exp\Big(-\frac{m|\boldsymbol{v}|^2}{2kT}\Big)\,,$$

with given m and T. (*Hint*: The distribution density ρ_v of v is *not* $\mathscr{N}\exp(-mv^2/2kT)$. Why not?)
(b) (5 points) Is $v_{\max}$ greater or less than $\sqrt{\mathbb{E}(\boldsymbol{v}^2)}$?
(c) (5 points) Determine $v_{\max}$ for N_2, the main constituent of air with $m = 4.6 \cdot 10^{-26}$ kg, at an absolute temperature of $T = 300$ Kelvin.

Hand in: Wednesday, May 8, 2019, in the exercise class.