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Mathematical Statistical Physics: Assignment 3

Problem 13: *Another proof of the belt theorem* (hand in, 30 points)
Let $\boldsymbol{X} = (X_1, \ldots, X_d)$ be a random, uniformly distributed point on $\mathbb{S}_1^{d-1}$. Use rotational and reflection symmetry to compute the expectation and variance of X_1. Then use the Chebyshev inequality to bound $\mathbb{P}(-\delta < X_1 < \delta)$ and prove the belt theorem as stated in Problem 5 on Assignment 1.

Problem 14: *A variant of the belt theorem* (hand in, 20 points)
Let $\boldsymbol{X}, \boldsymbol{Y}$ be independent, uniformly distributed random unit vectors in $\mathbb{R}^d$. Show that for every $\varepsilon, \delta \in (0,1)$ there is $d_0 \in \mathbb{N}$ such that, for all $d > d_0$, $|\boldsymbol{X} \cdot \boldsymbol{Y}| < \delta$ with probability at least $1 - \varepsilon$. ("In high dimension, independent vectors are nearly orthogonal.")
Hint: Conditionalize on $\boldsymbol{X}$ and use the belt theorem.

Problem 15: *Again near-orthogonality in high dimension* (don't hand in)
(a) In the setting of the previous problem, explain why $\mathbb{E}[(\boldsymbol{X} \cdot \boldsymbol{Y})^2] = 1/d$.

(b) Consider $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{S}_1^{d-1}$ with the "average" value $(\boldsymbol{x} \cdot \boldsymbol{y})^2 = 1/d$. Compute the angle α between $\boldsymbol{x}$ and $\boldsymbol{y}$ for $d = 3$ and asymptotically for large d (to leading non-constant order in d).

Problem 16: *Marginal of the uniform distribution on the sphere* (hand in, 50 points)
Let u_R^{d-1} be the uniform probability measure on the sphere $\mathbb{S}_R^{d-1}$ of radius $R > 0$ in $\mathbb{R}^d$, and let $(X_1, \ldots, X_d) \sim u_R^{d-1}$. Show that the marginal distribution of $X_1, \ldots, X_k$, $k < d$, has density given by

$$\rho_{k,d,R}(\boldsymbol{x}) = \frac{A_{d-k}}{A_d\, R^{d-2}}\, 1_{\boldsymbol{x}^2 \leq R^2} \left(R^2 - \boldsymbol{x}^2\right)^{(d-k)/2-1} . \tag{1}$$

Instructions: (a) Let $f_{k,d,R}(x_1 \ldots x_k)$ be the k-marginal of $1_{x_1^2 + \ldots + x_d^2 < R^2}$. Show that

$$\rho_{k,d,R}(\boldsymbol{x}) = R^{1-d}\, A_d^{-1}\, \frac{\partial}{\partial R} f_{k,d,R}(\boldsymbol{x}) \quad \text{for } |\boldsymbol{x}| < R. \tag{2}$$

(b) Let B_r^k denote the ball of radius r around the origin in $\mathbb{R}^k$. Show that for $0 < r < R$,

$$f_{k,d,R}(r, 0 \ldots 0) = \frac{1}{A_k\, r^{k-1}} \frac{\partial}{\partial r} \mathrm{vol}_d\Big((B_r^k \times \mathbb{R}^{d-k}) \cap B_R^d \Big) . \tag{3}$$

(c) Verify that

$$\mathrm{vol}_d\Big((B_r^k \times \mathbb{R}^{d-k}) \cap B_R^d \Big) = \int_0^r dr_1 \int_0^{\sqrt{R^2 - r_1^2}} dr_2\, r_1^{k-1}\, A_k\, r_2^{d-k-1}\, A_{d-k} \,. \tag{4}$$

(d) Conclude (5).

Problem 17: *Another way to compute the area of* $\mathbb{S}_1^{d-1}$ (don't hand in)
The integral

$$\int_{\mathbb{R}^d} d^d\boldsymbol{x}\; e^{-\boldsymbol{x}^2} \tag{5}$$

can be computed in two ways: as a product of d 1-dimensional integrals (whose values we know), or in spherical coordinates (where the angle integrals yield the area A_d of $\mathbb{S}_1^{d-1}$). Exploit this to show that

$$A_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}\,. \tag{6}$$

Hand in: Wednesday, May 15, 2019, in the exercise class.