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# Mathematical Statistical Physics: Assignment 5

**Problem 22:** *Identical particles and topology* (hand in, 30 points)
Let $\Gamma_1 = \mathbb{R}^6$. Show that ${}^N\Gamma_1$ is topologically non-trivial in the following sense: There are closed curves in ${}^N\Gamma_1$ that cannot in ${}^N\Gamma_1$ be contracted (i.e., continuously deformed to a point). Such a topological space is said to be non-simply connected.

**Problem 23:** *More about identical particles and topology* (don't hand in)
Let $\Gamma_1 = \mathbb{R}^d$. Show that $\Gamma_1^{N\neq}$ is simply connected, i.e., closed curves can be contracted, iff $d \geq 3$.

**Problem 24:** *Poincaré recurrence* (hand in, 40 points)
Let $\Omega = \mathbb{S}^1_1$ be the unit circle in $\mathbb{R}^2 = \mathbb{C}$, and let $T : \Omega \to \Omega$ be multiplication by $e^{i\alpha}$. Use the Poincaré recurrence theorem to show:

**(a)** $\forall \delta > 0 : \exists n \in \mathbb{N} : \forall x \in \Omega : \; d(x, T^n x) < \delta$,
where $d$ is the distance in arc length along the circle.

**(b)** For $\alpha \notin \pi\mathbb{Q}$ and every $x \in \Omega$, the set $T^{\mathbb{N}}x$ is dense in $\Omega$.

**Problem 25:** *Dense trajectory* (hand in, 30 points)
Let $\Omega$ be the 2-dimensional torus and $\varphi_1$ and $\varphi_2$ the angular coordinates on it (longitude and latitude). For given constants $\alpha_1, \alpha_2 \in \mathbb{R}$, consider the ODE

$$\frac{d\varphi_1}{dt} = \alpha_1\,, \quad \frac{d\varphi_2}{dt} = \alpha_2\,.$$

**(a)** Give an explicit formula for the flow map: $T^t(\varphi_1, \varphi_2) = \,?$

**(b)** Use Problem 24 to show that if $\alpha_2 \neq 0$ and $\alpha_1/\alpha_2 \notin \mathbb{Q}$, then the curve $t \mapsto T^t(\varphi_1, \varphi_2)$ is dense on the torus.

**Problem 26:** *Dense trajectory in higher dimension* (don't hand in)
Consider the corresponding situation on the $n$-dimensional torus $\mathbb{S}^1 \times \cdots \times \mathbb{S}^1$:

$$\frac{d\varphi_i}{dt} = \alpha_i\,, \quad i = 1, \ldots, n\,.$$

Under which condition on $(\alpha_1, \ldots, \alpha_n)$ is the curve $t \mapsto T^t(\varphi_1, \ldots, \varphi_n)$ dense on the torus?

**Problem 27:** *Recurrence times* (don't hand in)
In order to estimate the order of magnitude of realistic recurrence times, we reason as follows. An ideal gas comprising $N = 10^{23}$ particles (or a gas of $10^{23}$ hard spheres, the difference does not matter) in a box $\Lambda$ starts in such a phase point $x_0$ that all particles are located in the left half $\Lambda_L$ of the box; apart from that, let $x_0$ be typical of energy $E = N\bar{e}$; i.e., take $x_0$ to be a typical element of $M_L = \Gamma_E \cap (\Lambda_L^N \times \mathbb{R}^{3N})$. We want to know how long it takes, after $x(t)$ has left $M_L$, until $x(t)$ returns to $M_L$.

**(a)** Determine $\mu_E(M_L)$.

**(b)** Think of $\Gamma_E$ as partitioned into cells $C_1, \dots, C_r$ of equal volume (i.e., of equal measure $\mu_E$), of which $M_L$ is one. Assume that every cell gets traversed in time $\tau$, and that the trajectory $x(t)$ visits all cells in a random-looking order. How many years will pass before the return to $M_L$ if $\tau = 10$ s? If $\tau = 10^{-20}$ s?

**Hand in:** Wednesday, May 29, 2019, in the exercise class.