Prof. Dr. Roderich Tumulka | Summer semester 2019
Eberhard Karls University Tübingen | May 28, 2019

Mathematical Statistical Physics: Assignment 6

Problem 28: *Ergodicity on a finite set* (hand in, 35 points)
Let Ω be a finite set, $\#\Omega = n$. A dynamical system in discrete time means a bijection $T : \Omega \to \Omega$.
(a) Show that T preserves the uniform measure $\mathbb{P}(A) = \#A/n$.
(b) Show that $\mathbb{P}$ is the only probability measure preserved by every bijection.
(c) How many dynamical systems on Ω exist?
(d) What does it mean here for T to be ergodic?
(e) What are the ergodic components of a non-ergodic T?
(f) How many dynamical systems on Ω are ergodic?
(g) Determine the probability that a randomly chosen dynamical system on Ω is ergodic.
(h) The recurrence time of $\omega \in \Omega$ for T is defined as the smallest $t \in \mathbb{N}$ with $T^t\omega = \omega$. Determine the recurrence time for ergodic T.
(i) Determine the average recurrence time for random T (ergodic or not) and random ω.

Problem 29: *Scattering cross section for billiard balls* (hand in, 30 points)
When two billiard balls of radius a and momenta $\boldsymbol{p}_1, \boldsymbol{p}_2$ collide, the resulting (outgoing) momenta $\boldsymbol{p}'_1, \boldsymbol{p}'_2$ depend on the displacement vector $\boldsymbol{\omega} = (\boldsymbol{q}_2 - \boldsymbol{q}_1)/2a \in \mathbb{S}^2_1$ at the time of the collision:

$$\boldsymbol{p}'_1 = \boldsymbol{p}_1 - [(\boldsymbol{p}_1 - \boldsymbol{p}_2)\cdot\boldsymbol{\omega}]\boldsymbol{\omega}\,, \qquad \boldsymbol{p}'_2 = \boldsymbol{p}_2 + [(\boldsymbol{p}_1 - \boldsymbol{p}_2)\cdot\boldsymbol{\omega}]\boldsymbol{\omega}\,. \tag{1}$$

We consider *random* collisions and want to characterize the probability distribution of $\boldsymbol{p}'_1, \boldsymbol{p}'_2$ for given $\boldsymbol{p}_1, \boldsymbol{p}_2$ by determining that of $\boldsymbol{\omega}$. To this end, we suppose that $\boldsymbol{p}_2 = \boldsymbol{0}$ (as can be arranged via a Galilean transformation), $\boldsymbol{q}_2 = \boldsymbol{0}$, and $\boldsymbol{p}_1 = p_1\boldsymbol{e}_x$ (via translation and rotation). It is reasonable to assume that the y- and z-components of $\boldsymbol{q}_1$ are uniformly distributed on the disc of radius $2a$ around the origin in the yz-plane (given that a collision occurs at all); the polar coordinates r and φ of (y, z) are called the collision parameters.

(a) Express $\boldsymbol{q}_1$ and $\boldsymbol{\omega}$ as functions of r and φ.
(b) Show that $\boldsymbol{\omega} = (\omega_x, \omega_y, \omega_z)$ has probability density proportional to $1_{\omega_x<0}\,|\omega_x|$ relative to the uniform measure $u(d^2\boldsymbol{\omega})$ on the sphere.
(c) Explain why, for arbitrary $\boldsymbol{p}_1, \boldsymbol{p}_2$, the probability distribution of $\boldsymbol{\omega}$ is proportional to $1_{\boldsymbol{\omega}\cdot(\boldsymbol{p}_1-\boldsymbol{p}_2)<0}\,\big|\boldsymbol{\omega}\cdot(\boldsymbol{p}_1 - \boldsymbol{p}_2)\big|\,d^2\boldsymbol{\omega}$.

Problem 30: *Refrigerator* (don't hand in)
Use the second law of thermodynamics to show that a refrigerator necessarily consumes energy rather than generating energy. (We might have thought that the energy removed from the content of the refrigerator is available afterwards.)

Instructions. Let T_{in} be the temperature inside the refrigerator, T_{out} the one outside, Q_{in} the heat energy inside, Q_{out} the one outside, and δW the usable energy provided by the refrigerator (negative if it consumes energy) while it adds the energy $\delta Q_{\text{in}} < 0$ to the content and δQ_{out} to the outside. Use the Clausius relation $\delta S_i = \delta Q_i / T_i$, $i = \text{in}, \text{out}$ to show that $\delta W < 0$.

Problem 31: *Entropy in thermal equilibrium* (hand in, 35 points)
Compute the entropy $S(E, V, N)$ of the thermal equilibrium state of an ideal mono-atomic gas from Boltzmann's formula $S(\text{eq}) = k \log \Omega(E)$.

Instructions. We have already found the relations

$$\operatorname{vol}\Gamma_{\leq E} = \frac{1}{N!} V^N \, V_{3N} (2mE)^{3N/2} \tag{2}$$

$$V_d = \frac{\pi^{d/2}}{\Gamma(1 + d/2)} \tag{3}$$

$$\Omega(E) = \frac{d}{dE} \operatorname{vol}\Gamma_{\leq E} \, . \tag{4}$$

Use Stirling's formula

$$\Gamma(x+1) = \sqrt{2\pi x}\, e^{-x} x^x (1 + o(x)) \quad \text{as } x \to \infty \tag{5}$$

and $n! = \Gamma(n+1)$. Set $E = Ne$ and $V = Nv$ with constants e, v, sort terms by orders $O(N \log N), O(N), O(\log N), \ldots$, and give the leading order terms as the answer.

Hand in: Wednesday, June 5, 2019, in the exercise class.