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Mathematical Statistical Physics: Assignment 9

Problem 39: *ρ_t from the continuity equation* (hand in, 20 points)
Recall the continuity equation: If $X_0 \in \mathbb{R}^d$ is chosen randomly with (smooth) probability density $\rho_0 : \mathbb{R}^d \to [0, \infty)$ and $t \mapsto X_t$ is the solution of $dX_t/dt = F(t, X_t)$ with (smooth) time-dependent vector field $F : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$, then the probability density ρ_t of X_t obeys the continuity equation (5.14), i.e.,

$$\partial_t \rho_t(\boldsymbol{x}) = -\sum_{i=1}^{d} \partial_{x_i} \Big(\rho_t(\boldsymbol{x})\, F_i(t, \boldsymbol{x}) \Big) .$$

Now let F be the Hamiltonian motion on the phase space $\Gamma = \mathbb{R}^d$. Show that $t \mapsto \rho_t(X_t)$ is constant, in agreement with Problem 37(b) for $M = T^t$.

Problem 40: *Properties of the collision transformation* (hand in, 50 points)
At a collision of two billiard balls with collision parameter $\boldsymbol{\omega} = (\boldsymbol{q}_2 - \boldsymbol{q}_1)/2a$, the velocities change from $\boldsymbol{v} = \boldsymbol{v}_1$ and $\boldsymbol{v}_* = \boldsymbol{v}_2$ to

$$\boldsymbol{v}' = \boldsymbol{v} - [(\boldsymbol{v} - \boldsymbol{v}_*) \cdot \boldsymbol{\omega}]\boldsymbol{\omega} \qquad (1)$$

$$\boldsymbol{v}'_* = \boldsymbol{v}_* + [(\boldsymbol{v} - \boldsymbol{v}_*) \cdot \boldsymbol{\omega}]\boldsymbol{\omega} . \qquad (2)$$

Let $R_{\boldsymbol{\omega}}$ be the linear mapping $\mathbb{R}^6 \to \mathbb{R}^6$ with $R_{\boldsymbol{\omega}}(\boldsymbol{v}, \boldsymbol{v}_*) = (\boldsymbol{v}', \boldsymbol{v}'_*)$. Show that

- **(a)** $R_{\boldsymbol{\omega}}$ is orthogonal, $R_{\boldsymbol{\omega}} \in O(6)$. (*Hint*: By the polarization identity $\boldsymbol{u} \cdot \boldsymbol{v} = \frac{1}{4}\big(|\boldsymbol{u} + \boldsymbol{v}|^2 - |\boldsymbol{u} - \boldsymbol{v}|^2\big)$, it suffices for orthogonality of a linear mapping A that $|A\boldsymbol{u}| = |\boldsymbol{u}|$ for all $\boldsymbol{u}$.)
- **(b)** $\det R_{\boldsymbol{\omega}} = -1$. (You may use without proof that the determinant of a block matrix $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$, where A, B, C, D all commute with each other, is[1] $\det(AD - BC)$.)
- **(c)** $R_{\boldsymbol{\omega}}^2 = I_6$
- **(d)** $R_{-\boldsymbol{\omega}} = R_{\boldsymbol{\omega}}$
- **(e)** $\boldsymbol{\omega} \cdot (\boldsymbol{v}' - \boldsymbol{v}'_*) = -\boldsymbol{\omega} \cdot (\boldsymbol{v} - \boldsymbol{v}_*)$.

[1] J. R. Silvester: Determinants of Block Matrices. *The Mathematical Gazette* **84(501)**: 460–467 (2000)

Problem 41: *Boltzmann equation with external potential* (hand in, 30 points)
In an external potential V_1, the Boltzmann equation reads

$$\left(\frac{\partial}{\partial t} + \boldsymbol{v} \cdot \nabla_{\boldsymbol{q}} - \frac{1}{m} \nabla V_1(\boldsymbol{q}) \cdot \nabla_{\boldsymbol{v}}\right) f(\boldsymbol{q}, \boldsymbol{v}, t) = Q(\boldsymbol{q}, \boldsymbol{v}, t) \tag{3}$$

with the same collision term as given in the lectures,

$$Q(\boldsymbol{q}, \boldsymbol{v}, t) = \lambda \int_{\mathbb{R}^3} d^3\boldsymbol{v}_* \int_{\mathbb{S}^2} d^2\boldsymbol{\omega}\, 1_{\boldsymbol{\omega}\cdot(\boldsymbol{v}-\boldsymbol{v}_*)>0}\, \boldsymbol{\omega} \cdot (\boldsymbol{v} - \boldsymbol{v}_*) \times \Big[f(\boldsymbol{q}, \boldsymbol{v}', t)\, f(\boldsymbol{q}, \boldsymbol{v}'_*, t) - f(\boldsymbol{q}, \boldsymbol{v}, t)\, f(\boldsymbol{q}, \boldsymbol{v}_*, t)\Big] . \tag{4}$$

Show that the Maxwell-Boltzmann distribution is a stationary solution of this equation.

Problem 42: *A class of solutions of the Boltzmann equation* (don't hand in)
By comparing coefficients of powers of $\boldsymbol{v}$, show that functions of the form

$$f_t(\boldsymbol{q}, \boldsymbol{v}) = \exp\Big(A_t(\boldsymbol{q}) + \boldsymbol{B}_t(\boldsymbol{q}) \cdot \boldsymbol{v} + C_t(\boldsymbol{q}) \frac{\boldsymbol{v}^2}{2m}\Big) \tag{5}$$

with $A = A_1 + \boldsymbol{A}_2 \cdot \boldsymbol{q} + C_3 \boldsymbol{q}^2$, $\boldsymbol{B} = \boldsymbol{B}_1 - \boldsymbol{A}_2 t - (2C_3 + C_2)\boldsymbol{q} + \boldsymbol{B}_0 \times \boldsymbol{q}$, $C = C_1 + c_2 t + C_3 t^2$ are solutions of the Boltzmann equation without external field.

Hand in: Wednesday, July 3, 2019, in the exercise class.