Summer School on Paradoxes in Quantum Physics

Exercises

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Exercise 1. *Bohmian Trajectories for the Double Slit* (easy)

(a) Why do the Bohmian trajectories in Figure 1 not intersect? Give a mathematical reason.
(b) Why do they not cross the middle axis?

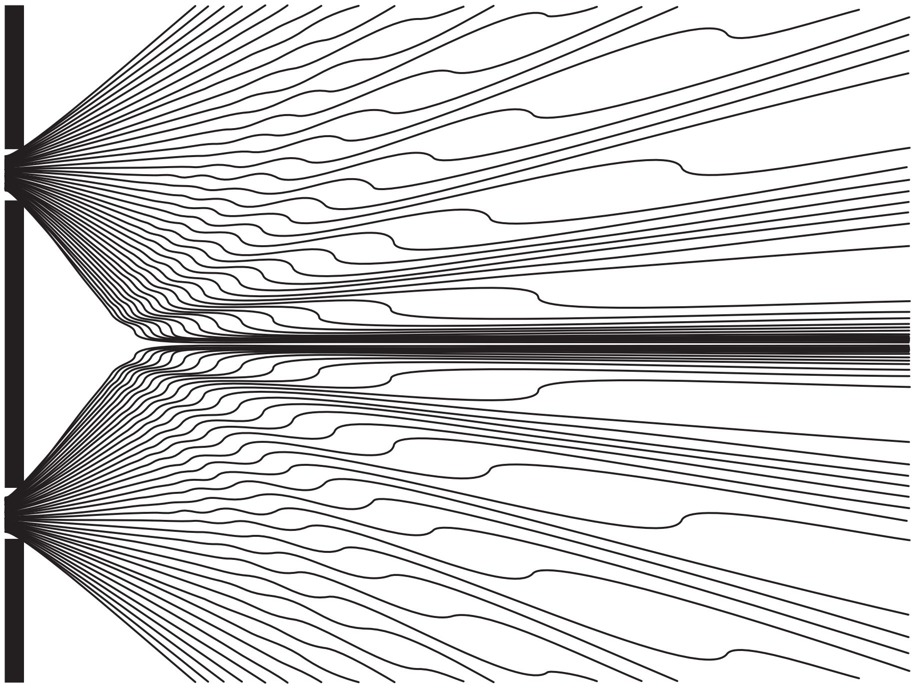


Figure 1: Bohmian trajectories for the double-slit experiment

Exercise 2. *Deviation of GRW Theory from QM* (easy)

According to GRW theory, a spontaneous collapse multiplies the wave function by a Gaussian factor of width σ. Consider the GRW theory with the constant σ much smaller than the value 10^{-7} m suggested by GRW; say, $\sigma = 10^{-12}$ m. Explain why Heisenberg's uncertainty relation $\Delta x\, \Delta p \geq \hbar/2$ implies that a free electron, after being hit by a GRW collapse, would move very fast. Use the uncertainty relation to compute the order of magnitude of how fast it can be (assuming it was more or less at rest before the collapse); the mass of an electron is about 10^{-30} kg and $\hbar \approx 10^{-34}$ kg m^2 s^{-1}.

Exercise 3. *Empirical Test of GRW Theory: Universal Warming* (intermediate)

Assuming that usual wave functions are wider than 10^{-7} m, a GRW collapse will tend to make a wave function narrower in position and, by the Heisenberg uncertainty relation, wider in momentum, thus increasing the average energy. As a consequence, all matter is spontaneously getting warmer all the time. Estimate the rate at which the temperature increases in Kelvin per year, given the parameter values suggested by GRW, $\sigma = 10^{-7}$ m and a collapse rate of $\lambda = 10^{-16}$ s^{-1} per nucleon. Use that an energy increase of ΔE corresponds to a temperature increase ΔT such that $\Delta E = \frac{3}{2} Nk\Delta T$ with N the number of molecules and $k \approx 10^{-23}$ J/K the Boltzmann constant. You may assume that a typical molecule contains of order 10 nucleons.

Exercise 4. *Probability Current* (easy)

Write $\psi(q) = R(q)e^{iS(q)/\hbar}$ with real-valued functions $R \geq 0$ and S. The probability current is defined as

$$j(q) = \tfrac{\hbar}{m}\mathrm{Im}\big[\psi^*(q)\,\nabla_q\psi(q)\big]\,.$$

Show that

$$j(q) = \frac{1}{m}R^2\,\nabla_q S\,. \tag{1}$$

Exercise 5. *Bohmian Mechanics is Time Reversal Invariant* (easy)

Show that if a trajectory $t \mapsto Q_t$ in configuration space $\mathbb{R}^{3N}$ is a possible history of Bohmian mechanics, i.e., a solution of Bohm's equation of motion

$$\frac{dQ}{dt} = \tfrac{\hbar}{m}\mathrm{Im}\frac{\nabla_q\psi(t,q)}{\psi(t,q)}\Big|_{q=Q_t} \tag{2}$$

with ψ a solution of Schrödinger's equation

$$i\hbar\frac{\partial\psi}{\partial t} = -\tfrac{\hbar^2}{2m}\nabla_q^2\psi(q) + V(q)\,\psi(q) \tag{3}$$

with real-valued potential $V(q)$, then the reverse history $t \mapsto Q_{-t} := \tilde{Q}_t$ is also a possible history, i.e., it is a solution of (2) with a different ψ obeying (3), viz., $\tilde{\psi}(t,q) := \psi^*(-t,q)$.

Exercise 6. *Pauli Matrices* (easy)

The three Pauli matrices are

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{4}$$

Verify that the following subsets of $\mathbb{C}^2$ are orthonormal bases consisting of eigenvectors of σ_1, σ_2, σ_3 with eigenvalues $+1$ and -1:

$$\left\{\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}\right\}, \quad \left\{\frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix}\right\}, \quad \left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right\}.$$

Exercise 7. *Disentangled Particles in Bohmian Mechanics* (easy)

Suppose a wave function of two non-interacting systems is disentangled for all times, $\psi(q_1, q_2) = \psi_1(q_1)\, \psi_2(q_2)$. Show that the Bohmian trajectory of system 1 does not depend on the initial condition of system 2.

Exercise 8. *Distinguish Ensembles* (intermediate)

In this variant of Bell's experiment, a source generates
(a) either 10,000 particle pairs in the spin singlet state
$\frac{1}{\sqrt{2}}\big(\big|\uparrow_z\downarrow_z\big\rangle - \big|\downarrow_z\uparrow_z\big\rangle\big) = \frac{1}{\sqrt{2}}\big(\big|\uparrow_x\downarrow_x\big\rangle - \big|\downarrow_x\uparrow_x\big\rangle\big)$
(b) or randomly distributed 5,000 pairs in $\big|\uparrow_z\downarrow_z\big\rangle$ and 5,000 in $\big|\downarrow_z\uparrow_z\big\rangle$
(c) or randomly distributed 5,000 pairs in $\big|\uparrow_x\downarrow_x\big\rangle$ and 5,000 in $\big|\downarrow_x\uparrow_x\big\rangle$.
Alice and Bob are far from each other, and each receives one particle of every pair. By carrying out (local) Stern-Gerlach experiments on their particles and comparing their results afterwards, how can they decide whether the source was of type (a), (b), or (c)?

Exercise 9. *Variant of Bohmian Mechanics* (intermediate)

In Bohmian mechanics, a particle is guided by a wave function ψ. We want to construct a variant of Bohmian mechanics in which a particle is guided by a density matrix ρ. The unitary evolution of the density matrix can be expressed by (assuming $\hbar = 1$)

$$\rho_t = e^{-iHt}\rho_0 e^{iHt} \text{ or } \frac{d\rho_t}{dt} = -i[H, \rho_t]\,.$$

A density matrix ρ can also be expressed as a function $\rho(q, q') = \langle q|\rho|q'\rangle$ with $|q\rangle$ the position (or configuration) basis.

(a) Set up an equation of motion analogous to Bohm's,

$$\frac{dQ}{dt} = \tfrac{1}{m}\mathrm{Im}\frac{\psi^*(q)\nabla_q\psi(q)}{\psi^*(q)\psi(q)}\Big|_{q=Q_t}, \tag{5}$$

but with the velocity determined by a density matrix $\rho(q, q')$ instead of a wave function $\psi(q)$.

(b) Verify that for a pure state, $\rho(q, q') = \psi(q)\, \psi^*(q')$, your equation of motion reduces to (5).

(c) (challenge) Show that the analog of Born's $|\psi|^2$ distribution, $\rho(q, q)$ is equivariant relative to your equation of motion $dQ_t/dt = v^\rho(Q_t)$,

$$\frac{\partial\rho(q, q, t)}{\partial t} = -\nabla_q \cdot \Big(\rho(q, q, t)\, v^{\rho_t}(q)\Big)\,.$$

Exercise 10. *No-Hidden-Variables Proof* (intermediate)

Verify the steps of the proof (due to Mermin and Perez) given below of the following no-hidden-variables theorem (due to Bell, Kochen, and Specker).

Theorem. There does not exist a function $v : \mathcal{A} \to \mathbb{R}$, where $\mathcal{A}$ is the set of self-adjoint matrices on $\mathbb{C}^d$ with $d \geq 4$, such that $v(-I) = -1$ and for all commuting $X, Y \in \mathcal{A}$,

$$v(XY) = v(X)\, v(Y)\,.$$

Proof. Regard a 4d subspace as $\mathbb{C}^2 \otimes \mathbb{C}^2$, and let σ_k^i be the Pauli matrix σ_k as in (4) acting on the i-th factor.

- $(\sigma_k^i)^2 = I$, $\sigma_x^i \sigma_y^i = -\sigma_y^i \sigma_x^i$, $\sigma_j^1 \sigma_k^2 = \sigma_k^2 \sigma_j^1$
- $\sigma_x^1 \sigma_y^2 \sigma_y^1 \sigma_x^2 \sigma_x^1 \sigma_x^2 \sigma_y^1 \sigma_y^2 = -I$ (move σ_x^1 from the first to fourth place, collecting a minus sign, then simplify)
- Define the self-adjoint matrices $A = \sigma_x^1 \sigma_y^2, B = \sigma_y^1 \sigma_x^2, C = \sigma_x^1 \sigma_x^2, D = \sigma_y^1 \sigma_y^2, E = AB, F = CD$.
- $EF = -I$
- $[A, B] = 0 = [C, D] = [E, F]$
- $-1 = v(-I) = v(EF) = v(E)v(F) = v(AB)v(CD) = v(A)v(B)v(C)v(D) = v(\sigma_x^1)v(\sigma_y^2)v(\sigma_y^1)v(\sigma_x^2)v(\sigma_x^1)v(\sigma_x^2)v(\sigma_y^1)v(\sigma_y^2)$
- The last expression is ≥ 0 since every factor appears twice. $\square$

Exercise 11. *POVMs* (challenge)

A discrete POVM (positive-operator-valued measure) is a finite collection of positive self-adjoint operators E_z on a system's Hilbert space $\mathscr{H}$ such that $\sum_z E_z = I$, the identity operator. The *main theorem about POVMs* asserts that *for every experiment that can be applied to a system with arbitrary wave function $\psi \in \mathscr{H}$ with $\|\psi\| = 1$, there exists a discrete POVM $\{E_z\}$ such that the probability of outcome z is given by $\langle\psi|E_z|\psi\rangle$.*

(a) Derive the main theorem about POVMs from GRWf, using that the joint distribution of flashes is given by a continuous POVM, i.e., a collection of positive operators G_f such that $\int df\, G_f = I$.

(b) Derive the main theorem about POVMs from Bohmian mechanics, assuming that the outcome of the experiment can be read off from the configuration of the apparatus at time T (at which the experiment is over).

Exercise 12. *Can't Distinguish Non-Orthogonal State Vectors* (intermediate)

(a) Alice gives to Bob a single particle whose spin state ψ is either $(1,0)$ or $(0,1)$ or $\frac{1}{\sqrt{2}}(1,1)$. Bob can carry out a quantum measurement of an arbitrary self-adjoint operator. Show that it is impossible for Bob to decide with certainty which of the three states ψ is.

(b) The same with only $(1,0)$ and $\frac{1}{\sqrt{2}}(1,1)$.

(c) (challenge) The same if Bob can carry out any experiment whatsoever. Hint: Use the main theorem about POVMs as described in Exercise 10.

Exercise 13. *The Mean King Problem*

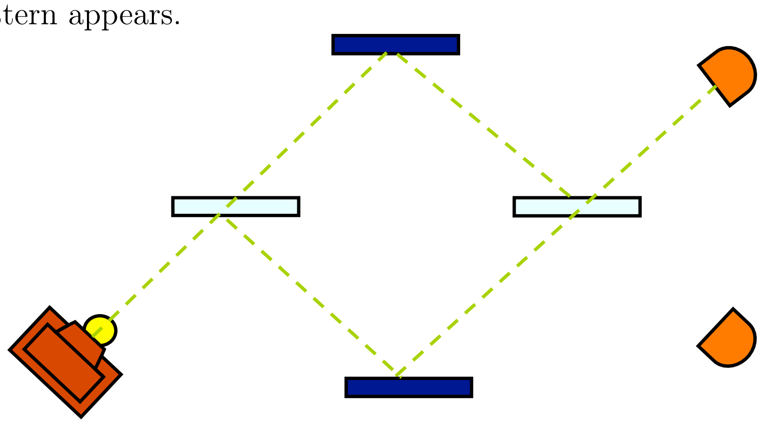
(a) (easy) A king asks you to prepare an electron on which he will perform a single quantum measurement of either σ_x or σ_z without telling you which measurement he did. After his measurement, he will give you the electron back, so you can perform your measurement on it. Your task is to retrodict with certainty the result the king got if he measured σ_x and the result he got if he measured σ_z. What should you do?

(b) (challenge) A king asks you to prepare an electron on which he will perform a single quantum measurement of either σ_x or σ_y or σ_z without telling you which measurement he did. After his measurement, he will give you the electron back, so you can perform your measurement on it. Your task is to retrodict with certainty the result the king got if he measured σ_x, the result he got if he measured σ_y, and the result he got if he measured σ_z. What should you do?

Exercise 14. *Ability to Interfere* (challenge)

(a) The Mach-Zehnder interferometer can be used with single photons. When a photon bounces off a mirror it transfers to the mirror the momentum $\sqrt{2}c^{-1}\hbar\omega$. By measuring the momenta of the mirrors, we can know which path the photon took. Explain why, nevertheless, an interference pattern appears.

(b) The Mach-Zehnder interferometer can also be used with single electrons. The electric field of an electron passing through one arm is different from that passing through the other arm. By measuring this field far away from the interferometer, we can know the path of the electron. Explain why, nevertheless, an interference pattern appears.



Exercise 15. *Versions of GRW Theory* (challenge)

GRWf and GRWm differ by the choice of primitive ontology. GRWf uses the GRW wave function ψ_t together with flashes at all collapse centers. GRWm uses the same wave function but claims that matter is continuously distributed with density

$$m(\mathbf{x},t) = \sum_{i=1}^{N} m_i \int d^3\mathbf{x}_1 \cdots d^3\mathbf{x}_N \, \delta^3(\mathbf{x}-\mathbf{x}_i) \, |\psi_t(\mathbf{x}_1 ... \mathbf{x}_N)|^2 \, .$$

Explain why GRWf and GRWm are empirically equivalent, i.e., make the same predictions.