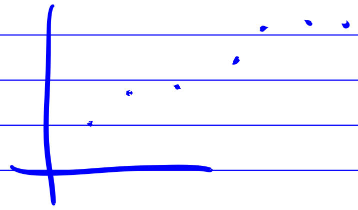
Überbestimmte Gleichungssysteme

Ausgleichsgerade: wir suchten Lösungen von

$$x_1 a \cancel{x_1} + 1 \cdot b = y_1$$
$$\vdots$$
$$x_m a \cancel{x_m} + 1 \cdot b = y_m$$



$$\begin{bmatrix} x_1 & 1 \\ \vdots & \vdots \\ x_m & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix}$$

für $m > 2$
i.d.R. widerspr.
$\Rightarrow \nexists$ Lsg.

Ebenso $Ax = b$, geg. A, $b \in \mathbb{K}^m$
A $m \times n$
ges. $x \in \mathbb{K}^n$

[Bsp: $x = \begin{pmatrix} a \\ b \end{pmatrix}$]

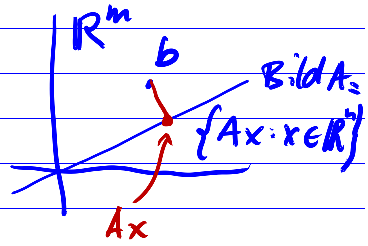
für $m > n$ i.d.R. überbest.
$\nexists$ Lsg.

"beste Approximation an eine Lsg."

minimiere $Q(x_1 \dots x_n) :=$

$$\sum_{k=1}^{m} \left(\sum_{j=1}^{n} a_{kj} \cancel{A} x_j - b_k \right)^2$$

$$= \| Ax - b \|^2$$



$$0 = \frac{\partial Q}{\partial x_i} = \frac{\partial}{\partial x_i} \sum_{k=1}^{m} \left(\sum_{j=1}^{n} a_{kj} x_j - b_k \right)^2$$

$$= \sum_{k=1}^{m} \frac{\partial}{\partial x_i} \left[\left(\qquad \right)^2 \right]$$

$$= \sum_{k=1}^{m} 2 \left(\qquad \right) \frac{\partial}{\partial x_i} \left(\qquad \right)$$

$$= \sum_{k=1}^{m} 2 \left(\sum_{j=1}^{n} a_{kj} x_j - b_k \right) \underbrace{\frac{\partial}{\partial x_i} \left(a_{ki} x_i \right)}_{a_{ki}}$$

$$= 2 \sum_{k=1}^{m} \left(\sum_{j=1}^{n} \underbrace{a_{ki}}_{(A^T)_{ik}} a_{kj} x_j - \underbrace{a_{ki} b_k} \right)$$

$$= 2(A^T A x)_i - 2\underbrace{(A^T b)_i}_{\sum_{k=1}^{m} \underbrace{(A^T)_{ik}}_{a_{ki}} b_k}$$

$$\Leftrightarrow \quad 0 = A^T A x - A^T b$$

$$\Leftrightarrow \quad \boxed{A^T A x = A^T b}$$ Gaußsche Normalengl.

$$\underbrace{\underset{(n \times m)}{A^T} \underset{(m \times n)}{A}}_{} \underset{(n \times 1)}{x}$$ $(n \times 1)$

$$\underbrace{\underset{(n \times m)}{A^T} \quad \underset{(m \times 1)}{b}}_{(n \times 1)}$$

$n < m$

Später: $\exists$ Lsg.

Bsp: Ausgleichsgerade

$n=2$

$x = \begin{pmatrix} a \\ b \end{pmatrix}$, $b = \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix}$, $A = \begin{pmatrix} x_1 & 1 \\ \vdots & \vdots \\ x_m & 1 \end{pmatrix}$

$$Ax = \begin{pmatrix} x_1 a + b \\ \vdots \\ x_m a + b \end{pmatrix}, \quad A^T A x = \begin{pmatrix} m\,\overline{x^2}\,a + m\bar{x}b \\ m\bar{x}\,a + mb \end{pmatrix}$$

$$= m \begin{pmatrix} \overline{x^2}\,a + \bar{x}b \\ \bar{x}\,a + b \end{pmatrix}$$

Normalengl. $A^T A x = A^T b \iff$

$$\cancel{m} \begin{pmatrix} \overline{x^2}\,a + \bar{x}\,b \\ \bar{x}\,a + b \end{pmatrix} = \cancel{m} \begin{pmatrix} \overline{xy} \\ \bar{y} \end{pmatrix}$$